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ELECTRICITY AND MAGNETISM

# MATHEMATICAL PHYSICS

VOLUME I.

# ELECTRICITY & MAGNETISM

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# MATHEMATICAL PHYSICS

VOLUME I.

# ELECTRICITY & MAGNETISM

BY

C. W. C. BARLOW, M.A., B.Sc.

JOINT AUTHOR OF "ELEMENTARY MATHEMATICAL ASTRONOMY"

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# PREFACE.

A SATISFACTORY knowledge of Physics must include Mathematics. Wide reading about apparatus, processes, and results is often rendered nearly worthless by the passage of a few years, which introduce new ideas and mechanism. But the mechanical interactions and numerical relations between physical quantities remain. The theory of the quadrant electrometer and the tangent galvanometer survives, when the construction of the instruments changes.

Every good text-book must therefore contain many numerical examples. This book, which is intended to supplement the ordinary text-book, is devoted exclusively to the mathematical aspect of the subject.

Only such descriptions of instruments are given as are essential to the type of instrument and are involved in its mathematical theory. General facts are considered from a numerical or geometrical point of view.

A special point has been made of the elementary mathematics of the electric discharge in vacuum tubes and of radioactivity. Enough is given to make some of the "popular" treatises really intelligible.

In an elementary text-book the mathematics is necessarily a little restricted. It has not been thought advisable to go beyond the rudiments of the Differential and Integral Calculus; and most of the work is frankly arithmetical. A great advantage of working out calculations in Electricity is that the student accustoms himself to deal accurately with quantities ranging over the widest diversity of magnitude.

Most of the numerical results used are taken from Kaye and Laby's *Physical and Chemical Constants* (1911), to which I am greatly indebted.

C. W. C. BARLOW.

*March* 1913.

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# CHAPTER I.

## POINT CHARGES.

**1. Order of Treatment.**—The earliest electrical experiment was to attract light bodies by rubbed amber. The laws involved in this apparently simple phenomenon are very complicated indeed. As our object is to apply mathematical principles, we are obliged to disregard both the historical order of electrical discovery and the most convenient order of experimental demonstration. We begin with such ideas as can be precisely defined and immediately subjected to mathematical processes, but which are simpler than the realities encountered in experiment. We assume that certain laws are obeyed. The consequences of these laws will prove to be exactly applicable to the ordinary bodies met with in experiments, such as electrified spheres, Leyden jars, magnets, batteries and coils. The order of thought is therefore outwards, from conceptions formed in the mind to objects visible in the world. It is taken for granted that these objects are already more or less familiar; so that it is unnecessary to aim at more than transforming vague acquaintance into exact knowledge.

**2. Point Charges.**—We assume that there exist **Charges of Electricity,** and that these can be localised on small portions of matter represented by gilt pith spheres in experiments. The air does not permit the escape of these charges and they can therefore be said to be **insulated.**

A charged body so small that its linear dimensions can be disregarded is called a **Point Charge.** The charge can be of either of two kinds, called Positive and Negative.

The magnitude of a charge can be called a **Quantity of Electricity.**

Experimental properties are that a positive and a negative point charge attract one another; but two like point charges, both positive or both negative, repel one another.

**3. Coulomb's Law.**—The force exerted by one point charge on another acts along the straight line joining them. It is proportional to the quantity of electricity in each, and to the inverse square of the distance between them. It is a repulsion if the charges are both positive or both negative; and an attraction if one is positive and the other negative.

If $q_1$ and $q_2$ be the charges, measured in any convenient units, and if $F$ be the force and $r$ the distance apart,

$$F \propto q_1 q_2/r^2$$

or

$$F = q_1 q_2/kr^2,$$

$k$ being a constant.

This law assumes that Electric Charge is a measurable quantity. The law itself provides a method of measuring charges, since every charge is proportional to the force it would exert on a given charge at a given distance. Both the laws and the assumption on which it rests are verified roughly by Coulomb's Experiment (§ 5), and with great accuracy by the Cavendish Experiment (§ 3). And, generally, the fundamental laws of any science are proved in the most absolute way by the agreement of complicated phenomena with the consequences deduced from these laws.

**Ex. 1.**—$A$, $B$, $C$, $D$ are small charged bodies. If $A$ be placed 6 inches from $B$ it repels it with a force of 20 (quite arbitrary) units. At the same distance $A$ repels $C$ with a force of 30 and $D$ with 35, and $B$ repels $C$ with 10. Find the forces which $B$ and $C$ would exert respectively on $D$ at the same distance apart.

Let $a$, $b$, $c$, $d$ be the small charges. The mutual repulsion between $A$ and $B = ab/kr^2$; but $r$ is the same in all the experiments, and therefore we can put the force $= mab$ units, $m$ a constant.

The equations are $mab = 20$, $mac = 30$, $mad = 35$, $mbc = 10$. We are asked to find the forces $x$ and $y$ which $B$ and $C$ exert on $D$. So put $mbd = x$, $mcd = y$.

Obviously $m$, $a$, $b$, . . . cannot be found separately. But

$$m^2abcd = mad \times mbc = 35 \times 10 = 350, \text{ and } mac = 30,$$

$\therefore$ $mbd = 350/30 = 11\frac{2}{3}$. Similarly $mcd = 350/20 = 17\frac{1}{2}$.

**Ex. 2.**—Two precisely equal small spheres have charges 5 and 19 arbitrary units; and attract each other, at distance 8 apart, with a force 20. They are allowed to touch. Assuming that the total charge is unaltered, but redistributed so that the spheres are charged equally, find the force exerted subsequently if the distance apart of the spheres be 5.

There is attraction, so the charges are unlike and may be called 19 and $-5$. The algebraic sum is 14 and $\therefore$ the charges will be 7 and 7 after contact. Coulomb's Law gives

$$20 = \frac{1}{k} \cdot \frac{5 \times 19}{8^2}, \quad \therefore \quad \frac{1}{k} = \frac{20 \times 8 \times 8}{5 \times 19} = \frac{256}{19}$$

in terms of the arbitrary units used.

After contact, the repulsion =

$$\frac{1}{k} \cdot \frac{7 \times 7}{5^2} = \frac{256}{19} \times \frac{49}{25} = 26{\cdot}4.$$

**Ex. 3.**—Two small charged spheres repel with a force of 24. After contact, they repel at the same distance with a force of 36. Find the ratio of their original charges.

**Ex. 4.**—Charges 8 and 10 attract with force 5 at distance 20. After contact, find the force with which they would repel at distance 4.

**4. Coulomb's Torsion Balance.**—For fuller description see any experimental treatise. We give only such details as will make the mathematical treatment intelligible.

The Torsional suspension is such as occurs in a large number of pieces of apparatus (Fig. 1). It consists of a fine vertical wire or fibre $AO$ of silver, phosphor bronze, quartz, or other suitable material. At its lower end it has an attachment or stirrup to receive the moving body, here a non-conducting lever $BC$ (Fig. 2). At the top it is supported clamped to a torsion head whereby its upper end can be rotated at pleasure, and the rotation can be read by graduations. These are usually degrees.

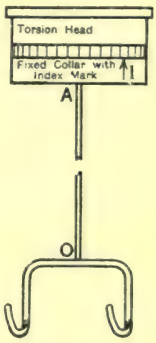


Fig. 1.

The lever $BC$ is very light, usually of shellac, and carries at one end, $C$, a small gilt pith ball

capable of holding an electric charge insulated. A second gilt pith ball, $D$, mounted on an insulating rod, can be set in a fixed position within the glass cylinder so that the rod is vertical and the balls $C$ and $D$ just touch in the zero position of the lever. The angular motion of the lever can be read by a scale concentric with it, and etched on a surrounding glass cylinder. When the position angle $DOC$ for the lever is zero, the torsion angle at $A$ should also be zero if the wire be without twist.

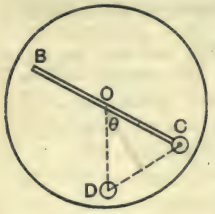


Fig. 2.

**5. Formula for Torsion Balance.**—To use the instrument, withdraw the ball $D$ and give a charge $2q$ to it. Replace $D$: the charge $2q$ is equally shared between $C$ and $D$, and repulsion takes place. Let the torsion head be now turned through an angle $\alpha$ so as to bring the balls nearer, and let the position angle $DOC$ of the lever be $\theta$. The lever is in equilibrium under

(1) A couple proportional to the angle of twist $\theta + \alpha$ of the wire, and denoted by $L(\theta + \alpha)$, where $L$ is a constant;

(2) The electrical repulsion $= q^2/CD^2$ acting in the straight line $CD$, and an equal and contrary force at $O$ which is the horizontal component of the tension in the wire. If $OC = OD = l$, these forces form a couple $lq^2 \cos \frac{1}{2}\theta/CD^2$;

(3) The weight of the bar and the equal and contrary vertical component of the tension; these balance. Hence the final equation of equilibrium is

$$L(\theta + \alpha) = lq^2 \cos \tfrac{1}{2}\theta/CD^2 = q^2 \cos \tfrac{1}{2}\theta/4l \sin^2 \tfrac{1}{2}\theta.$$

If, as usual, $\theta$ is small, we can put $\cos \frac{1}{2}\theta = 1$, $\sin \frac{1}{2}\theta = \frac{1}{2}\theta$, and the equation becomes $L(\theta + \alpha) = q^2/l\theta^2$. If the experimenter always turns the torsion head so far as to bring $\theta$ to a constant value, $q^2$ is proportional to $\theta + \alpha$, and therefore he is enabled to measure charges directly. If the instrument is to be used to prove Coulomb's Law,

the charges are left unaltered and the head turned to give a series of readings $\alpha$ for different readings $\theta$. Then, if $\theta$ be not large, $\theta^2(\theta + \alpha)$ should be constant. A different and much more accurate verification of the law is given in § 31.

**Ex. 1.**—A charge is given to the instrument, and $\theta$ is observed $= 90°$ when the torsion head is not turned. How much must it be turned to make $\theta = 60°$? Note that these values of $\theta$ are not small, and therefore the exact formula must be used.

**Ex. 2.**—$\theta = 15°$ when $\alpha = 0$. Find what values of $\alpha$ make $\theta = 10°$, and $5°$. Here the approximate formula may be assumed.

**Ex. 3.**—Let $\theta = 15°$ when $\alpha = 0$. The charge carried by the fixed ball $D$ is now removed, and therefore $C$ returns to touch $D$, and shares its charge, and is again repelled. Find what $\theta$ will be.

Here $q^2 \propto \theta^3$. Since $q$ is halved, $\theta$ is multiplied by $2^{-2/3}$. It becomes 9·4°.

**6. Absolute Units in Physics.**—The units used internationally are based on the centimetre, gram and second. It is stated (Chaney: *Our Weights and Measures*) that 1 inch = 25·39998 centimetres, and therefore = 2·54 to one part in a million. One lb. = 453·6 gm. When dealing with engineering problems one can use the rough approximations 1 kilometre = 5 furlongs = 5/8 mile; 1 mile = 1·6 kilometres; 1 ton = 1 tonne = 1,000 kilograms.

The unit of force is a dyne; this is defined as the force which would give unit acceleration to a particle whose mass is a gram. If $g$ be the acceleration of a body falling freely, the weight of one gram is $g$ dynes. The value of $g$ is 981 in London, Paris, and Berlin at the sea-level, and has very nearly the same value at all attainable localities.

The absolute unit of work is the erg, the work done by one dyne in moving its point of application one centimetre in its direction. The practical unit of work is the Joule, $= 10^7$ ergs. The absolute unit of power is the power of an agent which performs one erg per second. The practical unit of power is the Watt, = one joule per second.

**7. Electrostatic Unit Charge.**—The Unit of Charge or of Quantity of electricity is such a quantity that if it were collected at a point it would exert a repulsion of one dyne on an equal and similar quantity similarly collected, at a distance of one centimetre, provided that the surrounding medium is air at normal temperature and pressure.

If preferred, we can substitute vacuum for normal air in the definition. The practical difference is quite negligible.

The formula which expresses Coulomb's law is $F = \frac{q_1 q_2}{kr^2}$. But in air, using the above unit of charge, $F = 1$ when $q_1 = q_2 = 1$ and $r = 1$.

Hence $k = 1$ for air, and $F = q_1 q_2 / r^2$.

**Ex. 1.**—Two equal pith balls have masses ·10 gm. They hang from the same point by fine insulating cords 8 cm. long. A charge $2q$ is shared between them and they repel each other till equilibrium is reached at a distance 6 cm. apart. Find the absolute value of $q$.

The horizontal force on each $= q^2/6^2 = q^2/36$ dynes. If $\theta$ be the slope of either cord to the vertical, the resolved repulsion perpendicular to the cord $= q^2 \cos\theta/36$.

The weight of ball = 98·1 dynes and its resolved part = 98·1 $\sin\theta$.

These must be equal, $\therefore\ q^2 = 36 \times 98{\cdot}1 \times \tan\theta$.

But $6 = 2 \times 8 \times \sin\theta$, $\therefore\ \sin\theta = 3/8$, $\tan\theta = 3/\sqrt{55}$.

Hence $q^2 = 36 \times 98{\cdot}1 \times 3/\sqrt{55}$; whence $q = 37{\cdot}8$.

**Ex. 2.**—If $w$ = weight in grams of each pith ball, $q$ = the charge on each, $l$ = length of cord, and $\theta$ = angle which each cord makes with the vertical, prove that

$$q = 2l\sqrt{wg \sin^3\theta \sec\theta}.$$

**Ex. 3.**—For small deflections, if $x$ be the distance between the balls, show that approximately

$$q = x^{3/2}\sqrt{wg/2l}.$$

We deduce that if $x$ be the divergence (length or angle) between the leaves of a gold leaf electroscope, the charge is roughly proportioned to $x^{3/2}$.

**8. Field.**—An **Electric Field** is a region of space within which the influence of charged bodies extends.

We may speak of the field of one only of the bodies

considered, or of all of them, or of any number. Obviously the field of a point charge extends to infinity; but no influence is large enough to be detected at a considerable distance from a point charge.

**Field Strength, Intensity, Field** or **Force** at a given point of the electric field, is the force which a unit positive point charge would experience if placed at the point considered, if the electric distribution in its neighbourhood be supposed unaltered.

If air be the medium, the force which a charge $+ q$ would exert on unit charge at distance $r$ is $q \times 1/r^2 = q/r^2$. Hence the **field of a point charge** has magnitude $q/r^2$, and its direction is along the radius $r$.

**9. Composition of Fields.**—By definition, the fields at the same point, but due to two different distributions, are the forces which these distributions would exert on a unit charge at that point. They can therefore be compounded by the law of the Parallelogram or the Triangle of Forces. It follows that Field Strength is a vector quantity, representable in magnitude, direction and sense by a straight line, and obeying the ordinary vector laws.

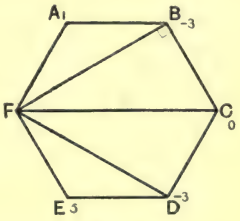


Fig. 3.

The following definitions are obvious:—

**Unit Field** is the field at a point at which a unit charge would experience a force of one dyne.

**Resolute of Field,** in any direction, is the resolute in that direction of the force which would be exerted on unit charge at the point considered.

**Ex. 1.**—$ABCDEF$ are the six angles of a regular hexagon. The charges at $A$, $B$, $C$, $D$, $E$ are $1$, $-3$, $0$, $-3$, $5$. Find the field at $F$; the side of the hexagon being 1 cm. (Fig. 3).

The distance $FB = FD = \sqrt{3}$; $FC = 2$.

The component fields are 1 and 5 in lines $AF$ and $EF$ produced, and $3/(\sqrt{3})^2 = 1$ and 1 in $FB$ and $FD$. Hence,

Sum of resolutes along $CF$ produced

$$= (1 + 5)\cos 60° - (1 + 1)\cos 30° = 3 - \sqrt{3}.$$

The sum of resolutes perpendicular to $CF$ in the direction of $EA = (5 - 1)\sin 60° = 2\sqrt{3}$.

Hence, if $F$ be the field, $F^2 = (3 - \sqrt{3})^2 + (2\sqrt{3})^2 = 24 - 6\sqrt{3} = 13{\cdot}608$. $\therefore F = 3{\cdot}70$.

Its direction makes an angle $\theta$ with $CF$ produced where $\tan\theta = 2\sqrt{3}/(3 - \sqrt{3}) = 1 + \sqrt{3} = 2{\cdot}732$. Hence $\theta = 70°$ nearly.

**Ex. 2.**—With the same data as in the above exercise, find the field at the centre of the hexagon.

**Ex. 3.**—If $ABCD$ is a square of 10 cm. side, and has charges 1, 1, − 1, − 1 at the corners in succession, find the field at the centre of the square and at the middle point of each side.

**10. Potential.**—The **Potential Difference (P.D.),** or excess of potential at one point over that at another point, is the work which the electric forces due to charges in the field would do on a unit positive point charge travelling (in air) from the first point to the second point. This unit charge is supposed not to alter the distribution of the actual charges in the field.

The **Potential** at a point is the P.D. at that point over a point of zero potential. We can arbitrarily choose the point of zero potential, but it is usual to take the potential as zero at any point infinitely distant from all the charges in the field.

In other media than air the same definitions hold if we imagine the unit +ve charge to travel from point to point along an infinitely narrow tunnel bored through the medium and occupied by air or vacuum.

The implied assumption that the work from point to point is independent of the path of the unit +ve charge is proved in the next paragraph. This work may be estimated for any one of the charges present, or for any selection of them, or for all. We can therefore speak of the potential at a point due to one, or some, or all of the charges in the field. And since the works done by different forces are additive, the potential at a point due to a group of charges is the algebraic sum of the potentials due to the charges separately.

**11. Potential Difference due to a Point Charge in air.**—Let $q$ be the charge at $O$; $ARSB$ any path from $A$ to $B$; $R$ and $S$ two points very near together. Drop perpendicular $RT$ on $OS$ (Fig. 4).

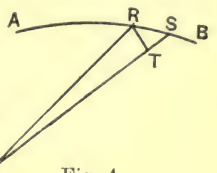


Fig. 4.

The force on unit charge at $S = q/OS^2$.

Its component along arc $RS$ is, in limit, equal to

$$q \cos RST/OS^2 = q \cdot \frac{ST}{RS} \cdot \frac{1}{OS^2}.$$

The work done by this force as the charge moves from $R$ to $S$

$$= RS \cdot q \cdot \frac{ST}{RS} \cdot \frac{1}{OS^2} = \frac{q \cdot ST}{OS^2}.$$

But in the limit $ST = OS - OR$; and $OS^2 = OS.OR$.

$\therefore$ The work from $R$ to $S = q \cdot \frac{OS - OR}{OS \cdot OR} = q \left\{ \frac{1}{OR} - \frac{1}{OS} \right\}$.

Adding such quantities for every small arc $RS$ we get, ultimately,

Potential difference between $A$ and $B$ = work from $A$ to $B$

$$= q \left\{ \frac{1}{OA} - \frac{1}{OB} \right\}.$$

It is evident that this depends only on positions of $A$ and $B$, and not on the form of the path $ARSB$. If $B$ be chosen as a position of zero potential, the potential at $A$

$$= q \left\{ \frac{1}{OA} - \frac{1}{OB} \right\}.$$

In the mathematical theory we choose $B$ to be at an infinite distance from every point charge in the field. Hence $OB = \infty$, $1/OB = 0$, and

Potential at $A = q/OA$.

We shall generally denote the potential by $V$; hence, if $r$ be the distance, $V = q/r$. The formula for media other than air is given in Chapter III.

**12. Composition of Potentials.**—If there be several point charges $q_1$, $q_2$, . . . , at $O_1$, $O_2$, . . . , the whole potential at $A$ (in air)

= work done by all of them on a unit charge travelling from $A$ to $\infty$,

$= q_1/O_1A + q_2/O_2A + q_3/O_3A + \ldots$

We can write this

$$V = \Sigma q/OA = \Sigma q/r$$

for any finite number (however great) of point charges.

And, taking the limit,

$$V = \int \frac{dq}{r}$$

for a continuous distribution, $dq$ being element of charge, and $r$ its distance from point $A$ considered.

**13. Equipotential Surfaces** are loci of points in space which have the same potential. Each surface can be specified by the common potential of all its points.

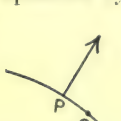


Fig. 5.

An **Equipotential Line** is the line in which an equipotential surface is cut by any surface considered,—usually a plane. The equipotential lines in a diagram are the lines in which the equipotential surfaces are cut by the plane of the diagram (*see* Figs. 6, 7).

If $P$ and $Q$ (Fig. 5) be indefinitely near points on an equipotential surface, the P.D. of $P$ and $Q$ is zero, and consequently no work would be done on a unit $+^{ve}$ charge travelling from $P$ to $Q$. Hence the force acting on this charge is perpendicular to $PQ$. That is, the field at $P$ (shown by arrow) is perpendicular to every indefinitely small arc $PQ$ on the equipotential surface, therefore the field at any point in space is normal to the equipotential through that point.

Evidently the equipotential surfaces of a point charge are a set of concentric spheres as in Fig. 6, and the field is everywhere radial.

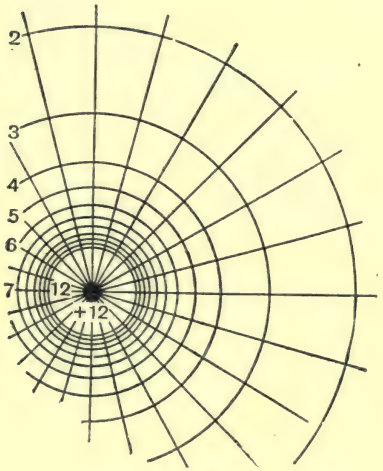



Fig. 6.

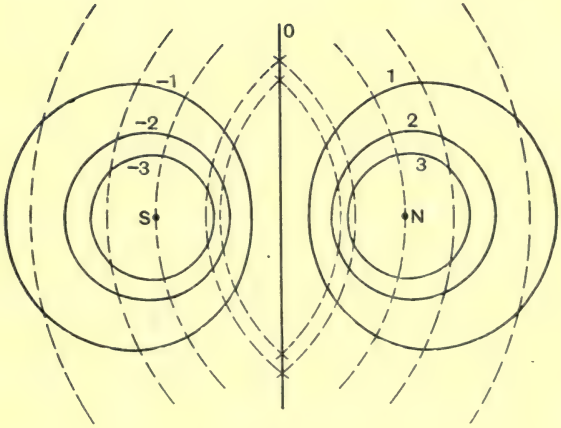



Fig. 7.

Fig. 6 shows the equipotential lines due to a charge + 12. Since $V = q/r$, $r = 12/2 = 6$ cm. when $V = 2$, $r = 4$ cm. when $V = 3$, and so on. [The figure is printed to a scale of $\frac{4}{7}$.]

**Ex. 1.**—Draw a set of equipotential lines for the charges 10 and − 10 placed at points $S$ and $N$ at 10 cm. apart. (Fig. 7 is drawn to a scale of $\frac{1}{3}$.)

In the figure the lines are drawn for the potential $V = 0, \pm 1, \pm 2, \pm 3$. If $P$ be a point at which the potential is + $V$, $10/NP - 10/SP = V$, or $10/NP = V + 10/SP$. The figure was constructed as follows :—

Describe about $S$ a series of circles of radii—

| | | | | | |
|---|---|---|---|---|---|
| $SP =$ | $7\frac{1}{2}$, | 8, | 10, | 12, | 15 |
| Then $10/SP =$ | 4/3, | 5/4, | 1, | 5/6, | 2/3 |
| If $V = 1$, $V + 10/SP = 10/NP_1 =$ | 7/3, | 9/4, | 2, | 11/6, | 8/3 |
| If $V = 2$, $V + 10/SP = 10/NP_2 =$ | 10/3, | 13/4, | 3, | 17/6, | 8/3 |
| If $V = 3$, $V + 10/SP = 10/NP_3 =$ | 13/3, | 17/4, | 4, | 23/6, | 11/3 |
| $\therefore NP_1 =$ | 30/7, | 40/9, | 5, | 60/11, | 6 |
| $NP_2 =$ | 3, | 40/13, | 10/3, | 60/17, | 4 |
| $NP_3 =$ | 30/13, | 40/17, | 5/2, | 60/23, | 30/11 |

For $V=1$, on each of the five circles of radius $SP$ touch two points whose distances from $N$ are the corresponding values of $NP_1$. Similarly for $V=2$, $V=3$. Of course $V=0$ gives the straight line bisecting $SN$ perpendicularly, The curves for $V$ negative are described like those for $V$ positive, interchanging $S$ and $N$.

**Ex. 2.**—With the data of the worked example, § 9, find the potentials at $F$ and centre of hexagon.

*Ans.* $(1 + 5)/1 + (- 3 - 3)/\sqrt{3} + 0/2 = 6 - 2\sqrt{3} = 2{\cdot}54$, and 0.

**Ex. 3.**—With the data of Ex. 2, § 9, find the potentials at the centre of the square and at the middle points of its sides.

**14. Lines of Force.**—A continuous line drawn in space whose direction at every point is that of the field at that point is called a Line of Force.

Since the resultant field at any point is perpendicular to the equipotential surface at that point (13), therefore lines of force cut equipotentials orthogonally.

Two lines of force cannot cut one another except at a point of zero or of infinite field. For wherever the field is finite its direction is determinate.

Similarly two different equipotential surfaces cannot

meet one another, for the potential cannot have two different values at one point.

Of course we are at liberty to consider the lines of force due to one only of the charged bodies, or to any selection. The lines of force or equipotentials due to different charge distributions can lie within the same region, and cut one another.

**15. Lines of Force of a Point Charge.**—Let $O$ be the point and $q$ the charge. The field at any point $P$ is in the straight line $OP$, and therefore the lines of force are straight lines radiating from $O$. Their number is infinite, but we shall classify them as $N$ unitary lines, where $N$ is an arbitrary number. The lines should be regarded as distributed equably or symmetrically in all directions around $O$.

To speak more precisely, the number of lines which pass through any area $A$ of a sphere, whose centre is $O$, must be proportional to $A$. The whole surface of a sphere of radius $r$ is $4\pi r^2$; hence each of our $N$ unitary lines should pass through an area $4\pi r^2/N$. Of course a "unitary line" is a group of lines, comprising an infinite number of individuals, which all pass through the same area $4\pi r^2/N$ described on the sphere.

The number of unitary lines per unit area is $N/4\pi r^2$. It is immaterial whether this be large or small, integral or fractional; for each unitary line, as above described, can be divided into as many equal submultiples as we like.

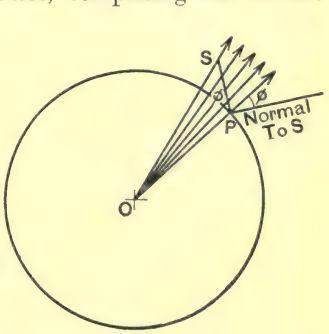


Fig. 8.

**16. Flux of Force.**—We define the Flux of Force, or of Field or Intensity, through any small area as equal to the number of unitary lines which cut through that area.

Consider the lines at $P$ of a point charge $q$ at $O$ (Fig. 8).

First let the area $A$ be very small, and perpendicular to the lines of force. We can regard $A$ as part of a sphere of radius $OP = r$; hence the flux through $A$ is $AN/4\pi r^2$.

But the field is $F = q/r^2$; therefore $1/r^2 = F/q$, and the flux is equal to $FAN/4\pi q$.

Next, let the area $S$ be oblique, making an angle $\phi$ with the area $A$ on spherical surface, so that $\phi$ is the angle between the field at $P$ and the normal to $S$. Then if $A$ be the projection of $S$ on the spherical surface, $A = S\cos\phi$, and every line which goes through one area goes through both. Hence

$$\begin{aligned}\text{Flux through } S &= \text{flux through } A \\ &= FAN/4\pi q = F\cos\phi.\ SN/4\pi q.\end{aligned}$$

The "flux per unit area" for $S = F\cos\phi \times N/4\pi q$, and is equal to the product of $F\cos\phi$, the normal component of the field at $P$, by the constant quantity $N/4\pi q$.

**17. Maxwell Tubes of a Point Charge. Faraday Tubes.**—In §§ 15, 16 the number $N$ was quite arbitrary. Two different conventions have been adopted to define it.

Considering, for the present, the field of a single point charge, we have seen that the flux per unit area through the small area $S$

$$= \text{Normal component of the field} \times N/4\pi q.$$

If we choose $N = 4\pi q$, our unitary lines may be called **Maxwell lines or tubes.** Hence

Resolute of Field in any direction

= Number of lines of force, per unit area, drawn perpendicular to that direction.

But $N$ is (§ 15) the whole number of lines originating from $q$; hence $4\pi q$ lines pass outward from charge $+q$.

But if we choose $N = q$, then our lines are called **Faraday Unit Tubes**; and we have the statements that $q$ Faraday tubes originate from charge $+q$; and that the field perpendicular to any small area = number of Faraday tubes per unit area $\times 4\pi$. The vector quantity

is often called the Electric Displacement. Consequently the Electric Displacement (normal) = number of Faraday Tubes per unit area.

**18. Surface Integrals.**—The Surface Integral of a vector quantity over a given surface is the sum of the products of every indefinitely small element of the area by the resolved part of the vector along the normal to the surface.

If $F$ be the vector, $dS$ the element of surface, and $\phi$ the angle between $F$ and the normal, then the surface integral is the result of integrating or summing all the products

$$F \cos \phi \,.\, dS.$$

If the vector $F$ be the resultant of $F_1$, $F_2$, $F_3$ . . . which makes angles $\phi_1$, $\phi_2$, $\phi_3$ . . . with the normal; then, resolving normally,

$$F \cos \phi = F_1 \cos \phi_1 + F_2 \cos \phi_2 + F_3 \cos \phi_3 \ldots$$
$$\therefore F \cos \phi \,.\, dS = F_1 \cos \phi_1 \,.\, dS + F_2 \cos \phi_2 \,.\, dS + F_3 \cos \phi_3 \,.\, dS. + \ldots$$

Therefore, summing,

> The Surface Integral of the resultant vector is the algebraic sum of surface integrals of the component vectors.

**19. Gauss's Theorems.**—(I.) The surface integral of the electric field taken over any closed surface which does not contain any charge is zero.

(II.) The Surface Integral of the electric field taken over any closed surface whatsoever = $4\pi$ . algebraic sum of the charges within the surface.

To prove these theorems, we first consider the surface integral of a point charge $+ q$ situated at any point $O$ (Fig. 9). Every product $F \cos \phi \,.\, dS$ = the number of lines of force from $O$ that proceed through $dS$. (§§ 16, 17).

Hence the sum of products for all elements $dS$ is the sum (algebraic) of all the lines from $O$ which pass outward through the whole surface $S$.

Lines passing inward must be taken with the contrary sign.

If $O$ be outside, every line from $O$ which cuts the surface must cut at least twice, to enter and pass out, and may cut any even number of times, entering as often as it leaves. Consequently the number of lines entering is exactly balanced by the number leaving, and the algebraic sum is zero.

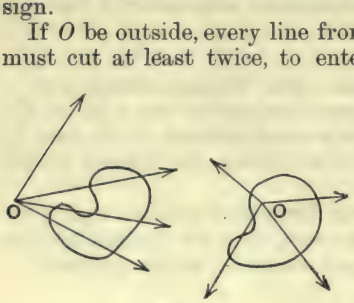


Fig. 9.

If $O$ be inside, every line from $O$ must cut the surface once to pass out, and, if it cuts oftener, the subsequent cuts are as often inward as outward, and exactly balance. Hence the algebraic sum is the number of lines of force which originate at $O$, $= 4\pi q$.

Next consider the surface integral of the field $F$ due to any number of positive or negative charges, $q_1$, $q_2$, $q_3$, etc. Let $F_1$, $F_2$, $F_3$, . . . be the fields due to $q_1$, $q_2$, $q_3$ . . . separately. Then the surface integral of $F$ = Algebraic sum of the surface integrals of $F_1$, $F_2$, $F_3$ . . . .

But if any of these be due to a point charge outside the surface, the corresponding surface integral is zero. And if any be due to a charge inside, the surface integral $= 4\pi \times$ that charge.

Hence, if all the charges be outside, the surface integral is zero; and, if otherwise, the surface integral $= 4\pi \times$ algebraic sum of the charges inside.

**20. Lines of Force due to any distribution.**— When we have more than one point charge, the lines of force are generally curved. Consider a bundle of lines of force starting from every point of an area $S_1$ and terminating on an area $S_2$ (Fig. 10). Consider the closed surface $S$ formed by $S_1$ and $S_2$ and the sides of the bundle, and let it contain no charge. The surface integral of field is zero over $S$ (§ 19); and it is evidently zero over the sides of

the bundle, for these are formed by lines of force and consequently the normal component of the force is zero.

Hence the surface integral, or flux, is equal, but of contrary sign, over $S_1$ and $S_2$.

If we divide the lines entering at $S_1$ into $n$ unitary tubes in such a way that $n =$ the flux inward over $S_1$, the same lines produced to $S_2$ will determine $n$ tubes leaving at $S_2$, and the number $n$ of lines will equal the outward flux over $S_2$.

Fig. 10.

Consequently, in any air space which contains no charges, we can suppose the lines of force drawn in such a manner that the number passing through any portion of any surface will equal the flux over that surface. And these lines, thus drawn, will be continuous in uncharged space. That is, none will originate, or terminate, within any region of air space in which there is no charge.

**21. Lines of Force originating from each Point-charge in an Electric Field.**—Let $S_1$ contain only one point charge $q$, and let $S_2$ be a sphere of small radius described

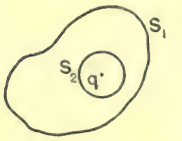


Fig. 11.

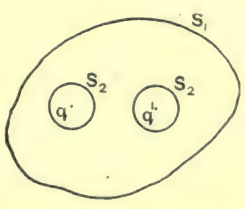


Fig. 12.

about $q$. If the radius $r$ of $S_2$ be taken small enough, the field on $S_2$ can be neglected except in so far as it is due to $q$, for $q/r^2$ is ultimately infinite compared to any other

term. Hence the number of unitary lines through $S_2$ is $4\pi q$; and none originate or terminate between $S_1$ and $S_2$; hence the number through $S_1$ is also $4\pi q$.

If $S_1$ contained more than one point charge such as $q$ and $q'$, $S_2$ could be regarded as a set of spheres of indefinitely small radius described about these charges, and we should have, by similar reasoning, that the total number of lines passing outward through $S_1 = 4\pi \times$ total charge inside. These lines are the general Maxwell lines for any distribution.

We could have used Faraday lines instead of Maxwell lines. In that case, the number of lines passing outward through $S_1 =$ total charge inside.

**22. Cross Section of Tube of Force.**—If Maxwell lines be taken, let $S_1$ and $S_2$ be portions of surfaces (equipotentials) orthogonal to the lines of force (Fig. 10, § 20), and take $S_1$, and therefore $S_2$, so small that the fields may be considered uniform over them of values $F_1$ and $F_2$. Then the equality of the surface integrals is expressed by $n = F_1 S_1 = F_2 S_2$. Hence $F S$ is constant for every orthogonal cross section of the same narrow bundle of lines. And $F = n/S$, that is—

> The field at any point = the number of Maxwell lines per unit area drawn at that point perpendicular to the field.

This is equal to $4\pi \times$ the number of Faraday tubes per unit area.

These theorems (from § 19 onward) are consequences of Coulomb's Law and would not hold in the case of any different law of radial force.

**23. Properties of Equipotential Surfaces.**—Let $S_1$, $S_2$ be two equipotentials for which the potentials $V_1$ $V_2$ differ very little. Let $P_1 P_2$ be an arc of an orthogonal line (line of force). Let $F$ be the average field between $P_1$ and $P_2$ so that the work done on unit + ve charge

travelling from $P_1$ to $P_2$ is $F.P_1P_2$. But this work $= V_1 - V_2$. Hence

$$F = (V_1 - V_2)/P_1P_2$$

or, the Field at any point = the Potential Gradient at that point, reckoned in the direction of the field.

The Potential Gradient is the limit of the fraction *P.D.*/Distance. It can be described as the Change of Potential per unit distance, or as the number of successive equipotentials (whose *P.D.* = 1) per unit length of line of force, or as the reciprocal of the distance between successive equipotentials. Observe in Figs. 6, 7 how the successive equipotentials approach one another in regions where the field is intense.

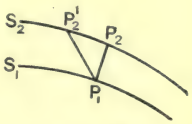


Fig. 13.

If next $P_1P_2'$ be not orthogonal, and $F'$ represent the average field component along $P_1P_2'$ between $P_1$ and $P_2'$, the work $= F' \times P_1P_2'$. Consequently the **resolved field $F'$ in any direction** = the potential gradient in that direction.

Note that these properties do not depend on the Law of Force (Coulomb's Law), but follow directly from the definition of potential.

**24. The Potential cannot be a maximum, or a minimum, at any point in empty space** unoccupied by a charge. For if it were a maximum, it would be possible to draw a closed surface around the point and so near it that the potential diminished along every outward normal. Hence the outward flux is everywhere positive, and the surface integral could not vanish. But this is impossible (§ 19). Similarly the potential could not be a minimum.

This property is deduced from Gauss's Theorem, which depends on Coulomb's Law. It would not be true under any other Law of Force. It is advisable to discriminate (in §§ 19-24) between theorems generally true and those which rest on the assumption of a special law of force.

**25. Regions of Constant Potential.**—(1) Within a closed equipotential surface, containing no charges, the potential is constant.

For otherwise the potential would be greater, or less, at some points than at others, and it would be possible to find a region of maximum or minimum potential; which is impossible (§ 19).

(2) If an equipotential surface, $A$, completely encloses another, $B$, at the same potential as $A$, the whole space between $A$ and $B$ is also at the same potential provided it contains no charge.

The proof is the same as for (1).

(3) If the potential be constant throughout any finite region of space, it is also constant throughout all space continuous with this region, provided it contains no charges.

For let $A$ be the region of constant potential. Let the potential be diminishing as we travel from $A$ within region $P$. Consider a closed surface enclosing part of $A$ and a narrow slice of $P$ lying along a short length of the boundary of $A$. The field is zero at every part of the surface within $A$, and is outwards within $P$, consequently the surface integral outwards is positive, and there must be a + charge within the surface, which is contrary to hypothesis. In the same way the potential cannot increase as we travel from $A$ in any region containing no charges. It therefore is constant.

# CHAPTER II.

## CHARGED CONDUCTORS.

**26. Conductors.**—A Conductor is any portion of a substance through whose interior an electric charge can flow. We assume that any charge tends to move in the same direction as that in which a charged body would be urged if situated at the same point.

It follows that **Positive Charges** in conductors **flow from regions of higher to regions of lower potential;** and negative charges flow the other way.

In a region of space within which electrical equilibrium is attained there can be no flow. Hence the potential must have a constant value throughout the substance of every conductor. What is the same thing, the **Field** within the substance of every conductor is zero when equilibrium is attained.

Also the boundary of each conductor must be an equipotential surface.

**27. The Electric Distribution on a Conductor is entirely superficial.**—We first show that the field cannot be zero, and therefore the potential cannot be constant, within any portion of space containing charges. For in a region containing charges, it is always possible to describe a closed surface containing more positive, or negative, electricity within it. The surface integral of the field outward over such a surface will have a finite value, positive or negative (§19). Hence the field is not everywhere zero, nor the potential constant.

But within the substance of a conductor in equilibrium the potential must be constant (§ 26). Hence there is no electrical distribution other than superficial.

This only applies, of course, to electricity at rest. When electricity is in motion, the flow is usually through the interior of the conductor.

**28. The potential is constant within a hollow conductor,** *i.e.* in a space completely enclosed, and containing no charges. For the boundary is an equipotential surface, containing no charges. Consequently (§ 25, 1) the potential is constant inside.

For the purposes of this chapter we shall ignore any difference (§ 68) that may exist between the constant potential within a hollow and the constant potential in the conducting substance. Both will be called the **Potential of the Conductor.**

If the conductor is charged, the charge must distribute itself over the outside surface in such a manner as to produce a constant potential within the substance of the conductor and within any hollow completely enclosed. If the conductor has zero charge, but there are charges outside, then charges must appear on the outer surface of the conductor whose algebraic sum is zero and which are so distributed as to make the total potential, due jointly to this distribution and the external charges, uniform within the conductor. This distribution is said to be Induced. If a conductor be charged and have charges outside, the distribution on the surface is everywhere the algebraic sum of the Induced charge and that distribution of the original charge which would, by itself, produce constant potential inside.

Since the potential is constant, the field is zero.

**29. Lines of Force and Conductors.—**

(1) No line of force penetrates within the substance of a conductor. For the field is zero.

(2) No lines exist in a cavity completely surrounded by conducting matter, which contains no charges.

(3) Lines of force meet a conductor normally; for they are always orthogonal to equipotentials.

(4) If we have a charge $+ q$ within a hollow conductor,

$4\pi q$ lines (or $q$ Faraday tubes) originate from it. These must terminate on the conductor, and consequently there must be a charge $-q$ on the inner conducting surface bounding the hollow. Thus a charge $+q$, within a hollow conductor, "induces" $-q$ on its inner surface. If the conductor be uncharged, $+q$ must appear on its outer surface. Hence a charge $-q$ inside a hollow conductor "induces" $+q$ on its inner surface and $-q$ on its outer surface. In these cases, the two induced charges are each numerically equal to the inducing charge.

(5) If a charge $+q$ be outside an insulated uncharged conductor, $4\pi q$ lines originate from it of which some terminate on it and some do not. If $4\pi p'$ terminate, a charge $-q'$ is induced; and $+q'$ appears on another part of the outside surface. Here $q'$ always $< q$, and the induced charges are numerically smaller than the inducing ones.

(6) If $\sigma$ be the actual surface density, or charge per unit area, on a conductor; and $F$ be the field just outside; then $F$ is normal and is equal to the number of lines of force per unit area. Hence $F$ lines originate from charge $\sigma$, therefore

$$F = 4\pi\sigma.$$

**30. Uniformly Charged Sphere.**—Assuming Coulomb's law, to show that the field within a uniformly charged sphere is zero.

Let $A$ be any point within the sphere. The chords $BAC$, $B'AC'$ indicate a double cone whose vertex is $A$ and whose solid angle $\delta\omega$ is indefinitely small (Fig. 14).

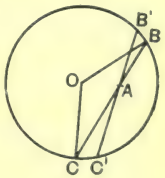


Fig. 14.

Let $\delta S_1$, $\delta S_2$, be the areas $BB'$, $CC'$ which the cone intercepts on the sphere. Let $O$ be the centre, and $a$ the radius.

Then $OB = OC$, therefore the angles $OBC = OCB = \phi$, say.

But since $\phi =$ the angle between the normal to $BB'$ and $AB$, therefore

$\delta S_1 \cos\phi =$ the projection of $BB'$ on to a plane at $B$ perpendicular to $AB$.

Hence $\delta S_1 \cos \phi / AB^2 =$ the solid angle subtended by $BB'$ at $A$, $= \delta\omega$.

Similarly $\delta S_2 \cos \phi / AC^2 = \delta\omega$, hence $\delta S_1 / AB^2 = \delta S_2 / AC^2$.

But the surface density $\sigma$ is given uniform, hence the charges on $\delta S$ and $\delta S_2$ are $\sigma.\delta S_1$ and $\sigma.\delta S_2$.

Therefore their repulsions on unit + charge at $A$ would be $\sigma.\delta S_1 / AB^2$ and $\sigma.\delta S_2 / AC^2$ by Coulomb's law.

But we have just shown that $\delta S_1 / AB^2 = \delta S_2 / AC^2$, hence these are equal and contrary.

This is true for every double cone with vertex at $A$.

Therefore the total field at $A$ is zero.

**31.** The **Cavendish Experiment** on Coulomb's Law.—If Coulomb's law be not precisely true, suppose that the nearer points exerted a larger repulsion in comparison with remoter points than Coulomb's law would give. Then, in figure, $BB'$ repels $A$ more than $CC'$, $\therefore$ $A$ is urged from the nearer to the remoter parts of sphere. By symmetry the resultant field at $A$ is along $AO$ towards $O$. Hence $O$ is at a lower potential than $A$, and therefore we have minimum potential at the centre of a positively charged sphere.

The contrary supposition that nearer points exerted a less repulsion would make the potential a maximum at the centre.

Experiments show that the potential is uniform and therefore there is neither a maximum nor a minimum; and the accuracy with which the experiment can be carried out measures the accuracy within which Coulomb's law is established with certainty.

It was shown by Maxwell that the force $\propto 1/r^k$, where $k$ does not differ from 2 by so much as $\pm$ 1/36000.

This method of proving Coulomb's law was due to Cavendish, who proved the law in 1771-1781, but with less accuracy than Maxwell attained. The great advances made by Cavendish were ignored and forgotten for nearly a century, when Clark Maxwell called attention to his researches after several of his results had been rediscovered

by others. Coulomb's own demonstration (§ 5) of the law of the inverse square is not susceptible of anything approaching the same accuracy.

**32. Uniformly Charged Sphere: External Field.**—To show that the field of a uniformly charged sphere at an external point is $Q/r^2$, where $r$ is the distance from the centre and $Q$ is the whole charge.

Let $D$ be the external point (Fig. 15), and $A$ be its inverse point {defined by the two relations that $OAD$ is a straight line and that $OA \,.\, OD = (\text{radius})^2$}. Make the same construction as in Fig. 14, and join $DB$. Then $OA \,.\, OD = OB^2$, hence $OA/OB = OB/OD$; therefore triangles $OAB$, $OBD$ are similar, hence $AB/BD = OB/OD = a/r$, and angle $BDO = OBA = \phi$.

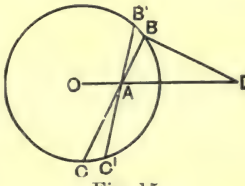


Fig. 15.

Then the element of surfaces $\delta S$ at $B$ has charge $\sigma\delta S$. The field at $D$ is $\sigma.\delta S/BD^2$ acting along $BD$ produced. The resolute along $OD$

$$= \sigma \,.\, \delta S \,.\, \cos\phi/BD^2 = \frac{a^2\sigma}{r^2}\,\delta S \cos\phi/AB^2.$$

But $\delta\omega = \delta S \,.\cos\phi/AB^2$ (§ 30), hence the resolute along $OD$ is $\dfrac{a^2\sigma}{r^2}\,\delta\omega$.

The resultant force due to the whole sphere acts, by symmetry, in direction $OD$. Hence its magnitude is $\Sigma\,\dfrac{a^2\sigma}{r^2}\,\delta\omega$, the sum of resolutes. But $\Sigma\,\delta\omega = 4\pi$; hence the field $= 4\pi a^2\sigma/r^2$. Now $\sigma = Q \div$ surface area $= Q/4\pi a^2$. Hence the field is $Q/r^2$.

**33. Potential of Uniformly Charged Sphere.**—The field is the same as that due to a point charge $Q$ located at the centre $O$. Hence if a unit +ve charge travel outside the sphere from infinity to any point $D$, the

work done on it is the same as if the charge of the sphere were concentrated at the centre. Hence the potential is the same. Its value (§ 11) is $Q/r$.

We can infer the potential within the sphere. If $D$ approach the surface, then ultimately $OD = r = a$, and the potential $= Q/a$. But the potential does not alter abruptly, hence it also $= Q/a$ on the surface of the conductor and within its substance and throughout any cavities completely enclosed which contain no charges.

**34. Field of an Infinite Plane Disc Uniformly Charged.**—In Fig. 16 let $DB$ meet the sphere again in $E$, and let $DBE$, $DB'E'$ indicate a cone of small solid angle whose vertex is $D$. As in § 30, the charges on $BB'$ and $EE'$ are proportional to $DB^2$, $DE^2$, hence the forces exerted at $D$ by these elements are equal.

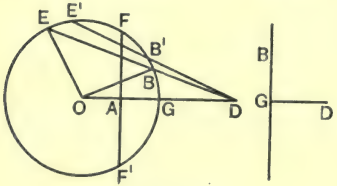


Fig. 16.

Hence if $FAF'$ be the plane which contains the point of contact of every tangent from $D$ to the sphere, this plane cuts the sphere into two portions which exert equal forces on $D$.

Hence the field due to the nearer portion $FGF' = 2\pi\sigma a^2/r^2$.

If the radius $a$ be indefinitely increased and the distance $GD$ of $D$ from the surface remains finite, the ratio $a^2/r^2$ will ultimately reduce to 1, and $FGF'$ becomes a flat disc. Hence the field due to a flat disc, uniformly charged, is $2\pi\sigma$.

**Ex. 1.**—Two small spheres have radii $a$ and $b$. Their centres are at a distance $c$ apart. Their charges are $x$ and $y$. Find the potential of each, assuming $c$ to be large compared with $a$ and $b$.

Let us call the spheres $A$ and $B$. The potential of $A$ due to its own charge is $x/a$. We can suppose the electrical distributions uniform, *i.e.* we consider $c$ so large that we may neglect induced electrification.

Then the potential of $A$ due to $B$'s charge is $y/c$. Hence the potential of $A$ is

$$x/a + y/c\,;$$

and similarly that of $B$ is

$$x/c + y/b.$$

**Ex. 2.**—If $A$ and $B$ be now connected by a fine insulated wire, find the charges and potentials of each.

As they are connected by a conductor, their potentials are equal; say $V$. If the charges be $x'$ and $y'$ we have

$$V = x'/a + y'/c = x'/c + y'/b,$$

and total charge is unaltered,

$$\therefore\quad x' + y' = x + y.$$

Solving,

$$x' = (x+y)\left(\frac{1}{b} - \frac{1}{c}\right)\Big/\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{c}\right)$$

$$y' = (x+y)\left(\frac{1}{a} - \frac{1}{c}\right)\Big/\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{c}\right)$$

$$V = (x+y)\left(\frac{1}{ab} - \frac{1}{c^2}\right)\Big/\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{c}\right).$$

**Ex. 3.**—Two small uncharged spheres whose centres are $A$ and $B$ and radii $a$ and $b$ are in the neighbourhood of a point charge $Q$ at $C$. Find their potentials. If the two spheres be connected by a fine wire, find their subsequent potential and charges.

Let $CA$, $CB$, $AB$ be $p$, $q$ and $c$. The potentials of $A$ and $B$ when uncharged are $Q/p$, $Q/q$.

When connected, let their charges become $x$ and $-x$. Then if the common potential be $V$ we have

$$V = Q/p + x/a - x/c = Q/q + x/c - x/b,$$

whence $x$ and $V$ are found.

**Ex. 4.**—Two small spheres, connected as above, are in a uniform field of magnitude H, whose direction makes an angle $\theta$ with $c$. Find the charges on each (Fig. 17).

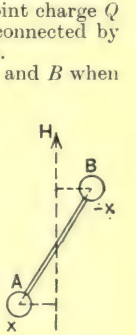


Fig. 17.

The field $H$, resolved along $AB$, is $H\cos\theta$; hence the potential difference (or P.D.) due to the field is $H\,c\cos\theta$. If the charges induced be $x$ and $-x$, then the total P.D. due to all causes

$$= H\,c\cos\theta + (x/a - x/c) - (x/c - x/b).$$

This has to be zero; hence $x$ is found. Neglecting $\frac{1}{c}$,

$$x = -\frac{H\,c\cos\theta}{\frac{1}{a} + \frac{1}{b}}.$$

**Ex. 5.**—Find approximately the couple exerted by the field on the system.

The forces exerted on the spheres are $Hx$, $-Hx$; and the distance between their line of action is $c \sin\theta$.

Hence the couple $= -Hx\,c \sin\theta$, which approximately

$$= H^2c^2 \sin\theta \cos\theta \Big/ \left(\frac{1}{a} + \frac{1}{b}\right).$$

**Ex. 6.**—If $I$ be the moment of inertia of the system about the centre of the wire, find the period of small oscillation (angular).

The couple is ultimately $= L\theta$, where $L = H^2c^2 \Big/ \left(\frac{1}{a} + \frac{1}{b}\right)$, if $\theta$ be small. This $= I\alpha$ if $\alpha$ be the angular acceleration.

But we know from mechanical principles (§§ 181, 184) that if $\alpha/\theta$ be constant the motion is angular simple harmonic motion of period $T$ where

$$T = 2\pi\sqrt{\theta/\alpha}.$$

Hence

$$T = \frac{2\pi}{Hc}\sqrt{\frac{I\,(a+b)}{ab}}.$$

Note that this experiment gives a means of measuring horizontal electric fields. The apparatus consists of gilt pith balls connected by a fine wire and suspended by a silk thread as in Figure 18. The field $H$ is proportional to the first power of the frequency; and obviously the direction of the field is indicated by the direction of the line joining the balls when in equilibrium. Any elongated conductor, such as a thick wire with rounded ends, will indicate the field in a similar manner; but the mathematical theory is more difficult.

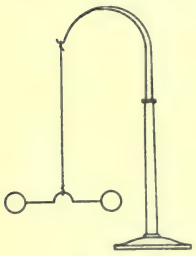


Fig. 18.

**Ex. 7.**—Three small equal spheres of radius $r$ have their centres at the corners of a triangle whose sides are $a$, $b$, $c$. Find the ratio of their charges that all may be at the same potential.

**Ex. 8.**—Two spheres of 1 cm. diameter are connected by a fine wire (of negligible capacity) so as to be 3 cm. apart. If they be originally uncharged and then put in a uniform electric field of unit strength, calculate the charge on each when the line joining their centres is in the direction of the field.

**35. Electric Images.**—If two systems of charges $C$ and $C'$ would give the same potential at every point of

a finite closed surface $S$, which contains no charge belonging to either of them, they also give the same potential at every point within $S$ (Fig. 19).

For let $-C'$ denote the distribution produced by reversing every charge in $C'$, then $C$ and $-C'$ together produce zero potential at every point of $S$; and therefore (§ 25) at every point within $S$. Hence, within $S$, $C$ and $C'$ give the same potential.

Similarly, if $C$ and $C'$ give the same potential at every point of a closed surface $S$, outside of which no charge belonging to $C$ and $C'$ lies (Fig. 20), they give the same

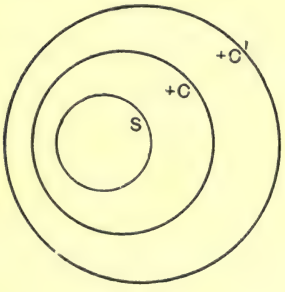



Fig. 19.

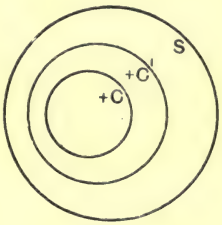



Fig. 20.

potential everywhere outside $S$. For $C$ and $-C'$ produce zero potential both over $S$ and at infinity, and therefore at all points between (§ 25).

If two such distributions produce the same potential throughout an extended region of space, they also produce the same force or field (§ 25). They can be called equivalent distributions for the region considered, and either is called the Image of the other.

In the second case, the potential due to $C$ and $-C'$ is zero everywhere outside $S$. Hence the field outside $S$ is also zero, and therefore the Surface Integral (§ 19) is zero. Consequently the total charge of $C - C'$ is zero, therefore the total charges of the equivalent systems $C$ and $C'$ are the same. This need not be true in the first case.

**36. Use of Electric Images.**—If a point charge $Q$ be outside a hollow conductor, kept at zero potential, there will be induced an electric distribution on the conductor which we may call $-X$. Let $S$ be the surface of the conductor (Fig. 21). Then its potential must be zero, therefore $Q$ and $X$ produce the same potential everywhere on $S$ and everywhere in its interior. By previous paragraph $Q$ is the image of $X$ with respect to all inside points.

If a point charge $q$ be inside a hollow conductor kept at zero potential, let the charge induced on the inner surface

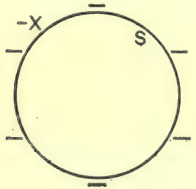



Fig. 21.

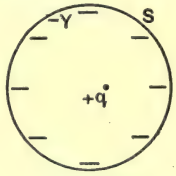



Fig. 22.

be called $-Y$. Then $q$ and $Y$ are equivalent both for the inner surface and all points outside; hence $q$ is the image of $Y$ for external points (Fig. 22). The total amount of $Y$ is $q$ (§ 35).

In both these cases a possibly complicated distribution $X$ or $Y$ has a simple image, namely a point charge $Q$ or $q$.

**37. Earthed* Plane Conductor.**—A point charge $Q$ is put at $A$, in front of infinite plane $CE$ at distance $c$. Drop $AC$ perpendicular to the plane, and produce to $B$, so that $AC = CB = c$ (Fig. 23).

If, as in § 36, the induced distribution be called $-X$, $X$ is equivalent to $Q$ at $A$ for all points behind the plane

* "Earthed" means kept at zero potential. Although the potential of the earth is not precisely zero in the mathematical sense, nor even precisely constant, it is small enough to be taken as zero for most practical purposes. A good earth connection is a metallic connection to a moist stratum; *e.g.* a water pipe.

$CE$. By symmetry, $X$ is also equivalent to $Q$ at $B$ for all points in front of the plane.

At any point $D$, the potential is due to $Q$ at $A$ and to $-X$. If $D$ is in front, as drawn, this is equivalent to $Q$ at $A$ and $-Q$ at $B$. Hence the potential

$$= Q\{1/AD - 1/BD\}.$$

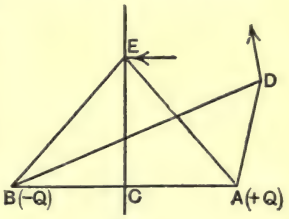


Fig. 23.

In the same way, replacing $-X$ by the image $-Q$ at $B$, the field at $D$ is the resultant of $Q/AD^2$ along $AD$ produced and $Q/BD^2$ along $DB$. At $E$, quite close to the surface, the forces $Q/AE^2$, $Q/BE^2$ are equal and equally inclined to $CE$. Their resultant is perpendicular to $CE$, as it should be (§ 29). Its magnitude, got by resolving in the direction $BA$,

$$= 2.\frac{Q}{AE^2}\cdot\frac{AC}{AE} = 2\ cQ/AE^3.$$

If $\sigma$ be the surface density induced on the plane conductor at $E$, the outward field $= 4\pi\sigma$ (§ 19). Hence

$$\sigma = -\ cQ/2\pi\ .\ AE^3.$$

Thus the induced distribution $-X$ has a density proportional to the inverse cube of the distance from $A$.

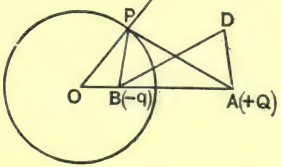


Fig. 24.

**38. Earthed Sphere. Charge Outside.**—Let $a$ be the radius of the sphere, centre $O$ (Fig. 24). Let $Q$ be a point charge at $A$ at a distance $OA = r, r > a$. On $OA$ take a point $B$ so that $OA.\ OB = a^2$; $OB = s = a^2/r$. Then $A$ and $B$ are inverse points with respect to the sphere.

If $-X$ be the induced distribution on the sphere, $Q$ at

$A$ and $-X$ produce zero potential on and within the sphere, and $Q$ is the image of $X$ for all inside points.

Take any point $P$ on the sphere; then by geometry $PA/PB$ = a constant $= r/a = a/s$. A charge $-q$ at $B$, together with $Q$ at $A$, would produce at $P$ the potential

$$Q/AP - qB/P$$

which is zero if $q = Qa/r$. In this case, $q$ at $B$ is equivalent to $Q$ at $A$ everywhere on the surface of the sphere. And $Q$ at $A$ is equivalent to $X$ for such points. Hence also $q$ at $B$ is equivalent to $X$ for such points.

Hence $q$ at $B$ is the image of $X$ for all external points. And (§ 35) the total amount of $X$ is equal to $q$; therefore the charge induced by $+ Q$ on the earthed sphere is $-q = - Qa/r$.

The potential at any outside point $D$ is due to $Q$ and $-X$, which is the same as $Q$ and $-q$. Its value is $Q/DA - q/DB$.

The field at $D$ would be the resultant of $Q/DA^2$ along $AD$, and $q/DB^2$ along $DB$. The field at $A$ due to $- X$ is $q/AB^2$, hence the force exerted on $Q$ is

$$Qq/AB^2 = Qq/(r - a^2/r)^2.$$

But $q = aQ/r$, hence the force $= Q^2ar/(r^2 - a^2)^2$.

**39. Induced Surface Density.**—The field at $P$ is the resultant of $Q/AP^2$ along $AP$, $q/PB^2$ along $PB$. If $OPN$ be the normal,

Tangential component of field

$$= Q \sin APN/AP^2 - q \sin BPO/BP^2$$

$$= \frac{Q}{AP}\left\{\frac{\sin APN}{AP} - \frac{\sin BPO}{BP}\right\}, \text{ since } \frac{Q}{AP} = \frac{q}{BP}.$$

But

$$OB : OP = OP : OA, \therefore \triangle s\ OBP, OPA \text{ are similar,}$$
$$\therefore \angle APN = 180° - APO = 180° - OBP = ABP$$

and

$$\angle BPO = PAB,$$
$$\therefore \sin APN/AP = \sin BPO/BP.$$

Hence the tangential component is zero, as it should be (§ 29).

The normal component of field

$$= Q \cos APN/AP^2 + q \cos BPO/BP^2$$

$$= \frac{Q}{AP}\left\{\frac{\cos APN}{AP} + \frac{\cos BPN}{BP}\right\}$$

$$= \frac{Q}{AP}\,\frac{BP \cos PBA + AP \cos PAB}{AP \,.\, PB}$$

$$= \frac{Q \,.\, AB}{AP^2 \,.\, PB}.$$

But $AB = r - s = r - a^2/r$, $PB = AP.\ a/r$. Hence the field at $P$ is normal inwards, and its value is

$$Q\,(r^2 - a^2)/a.\ (AP)^3.$$

If $\sigma$ be the surface density, the field is $4\pi\sigma$,

$$\therefore\ \sigma = -\,Q\,(r^2 - a^2)/4\pi a.\ AP^3.$$

Hence it varies as the inverse cube of the distance from the inducing point.

**40. Insulated Sphere. Charge Outside.**—Let $Q$ be the point charge at $A$, and $U$ be original charge on the insulated sphere.

If we construct as in § 38, and imagine a charge $-\,q = -\,Qa/r$ at $B$, this $-\,q$ would be the image, for space outside the sphere, of the distribution $-\,X$ described in § 38. Also $Q$ is the image, for the space inside the sphere, of $X$. Consequently $Q$ and $-\,X$ would produce zero potential throughout the interior of the sphere.

The total quantity of $-\,X$ is $-\,q$; so if the distribution $-\,X$ were formed on the sphere there would be an extra charge $U + q$. This must produce constant potential $(U + q)/a$ within the sphere, and therefore is uniformly distributed, and acts outside the sphere like a charge $U + q$ concentrated at $O$.

Now the actual charge distribution has to satisfy the two conditions that it maintains constant potential within the sphere and that its total quantity is $U$. These condi-

tions are satisfied by $-X$ distributed according to the law previously found (§ 39) and $U+q$ distributed uniformly. Hence these form the image of the induced charge. Hence the resulting charge is made up of two parts,

(1) The original $U$, distributed uniformly, and equivalent (for external points) to a point charge $U$ at the centre.

(2) A distribution $Z$ of zero quantity and equivalent (for external points) to two equal and contrary charges $q$ and $-q$ located at $O$ and $B$.

Two such charges, equal, contrary, and separated by a finite distance, are called a **Doublet.** Its **Moment** is the product of either charge by the distance between. Hence the image of the distribution $Z$ is the doublet

$$q \,.\, OB = qa^2/r = Qa^3/r^2.$$

**41. Density on uncharged Insulated Sphere.**—Let us put $U = o$. The normal field is as found in § 39, with the addition of $q/a^2$ outwards, due to the image charge $q$ at $O$. Hence the density is

$$\sigma = -Q\,(r^2-a^2)/4\pi.\,AP^3 + q/4\pi a^2$$
$$= \frac{Q}{4\pi}\left\{\frac{r^2-a^2}{AP^3} - \frac{1}{r}\right\}.$$

The force exerted by an uncharged sphere on a point charge $Q$ is an attraction,

$$= Qq/AB^2 - Qq/OA^2$$
$$= Q^2ar/(r^2-a^2)^2 - Q^2a/r^3 = Q^2a^3(2r^2-a^2)/r^3(r^2-a^2)^2.$$

This is infinite when $r = a$.

The force exerted by a sphere with charge $U$ is the resultant of this attraction and the repulsion $QU/r^2$. If $r$ is nearly $= a$, the resultant is an attraction even if $Q$ and $U$ have the same sign.

**42. Uncharged Insulated Sphere in Uniform Field.**—Let the field be $F$. We can suppose it due to an infinite charge $Q$ at an infinite distance $r$, so that $Q/r^2$ is finite and $= F$.

Then the induced distribution is due to the doublet of moment $Qa^3/r^2 = Fa^3$.

Its length $OB = s = a^2/r$ is, of course, infinitely small, and its charges $\pm q$ are infinitely great.

The normal force at $P$ due to the doublet is the limit of $q/PB^2 - q/PO^2$ inwards. If $< BOP = \theta$, thus ultimately

$$\frac{q}{(a - s\cos\theta)^2} - \frac{q}{a^2} = \frac{2qs\cos\theta}{a^3}, \text{ neglecting } s^2.$$

Put for $qs$ its value, the above $= 2F\cos\theta$. Hence the resultant normal force due both to the doublet and the inducing field $F$ is $3F\cos\theta$, and the density is $\sigma = 3F\cos\theta/4\pi$.

**43. Earthed Hollow Sphere. Charge Inside.**—Let the charge inside be $+Q$ at $A$ at distance $s$ $(s < a)$, and the induced distribution be called $-Y$, the other letters being as in § 38. Then $Q$ and $-Y$ produce zero potential over the inner surface, therefore $Q$ is the image of $Y$ for all outside points, and the total charge of $Y$ is $Q$ (Fig. 25).

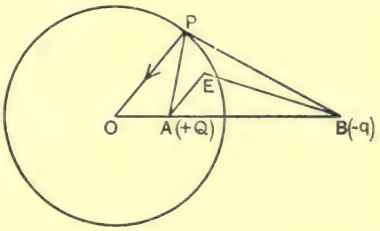


Fig. 25.

Imagine $q$ at $B$, where $q\ Q = r/a = a/s$. Then $q$ and $Q$ produce the same potential at every point of the surface. Therefore so do $q$ and $Y$. Hence $-q$ is the image of $-Y$ for all internal points.

Hence the field at $E$ anywhere within the hollow is the resultant of $Q/EA^2$ along $AE$ produced, and $q/EB^2$ along $EB$. And the potential at $E$ is $Q/EA - q/EB$,

$$= Q\{1/EA - a/s\ EB\}.$$

The field at $P$ just on the surface can be shown, as in § 39, to be normal and to have the value $q(r^2 - a^2)\ a.AP^3$.

This $= Q\ (a^2 - s^2)/a.\ BP^3$, if we replace $q$, $r$, $AP$ by $Q\ a/s$, $a^2/s$, $BP.a/s$. Hence the surface density is

$$-\ Q\ (a^2 - s^2)/4\pi a.\ BP^3.$$

The resultant force on $q$ at $B$ is, as in § 38,

$$= Qq/AB^2 = Q^2\ as/(a^2 - s^2)^2.$$

**44. Uniformly Electrified Rod or Cylinder.**—In Fig. 26 $O$ represents the axis. The field at any point, $P$, is (by symmetry) radial and a function of $OP$ only. Let it be $F$. Let $OP = r$, and radius of cylinder $= a$. Take the surface integral over a cylinder coaxial with the given cylinder, of radius $r$ and length $h$. Its curved surface is $2\pi rh$, and the surface integral is $2\pi rhF$; since evidently there is no surface integral over the flat ends.

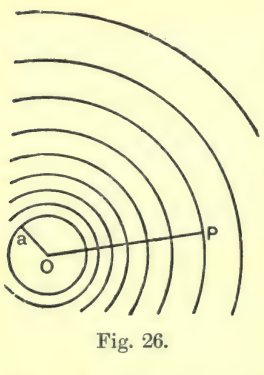


Fig. 26.

Consequently $2\pi rhF = 4\pi \times$ charge inside.

If $r > a$, as in figure, let $\sigma =$ the surface density on the cylinder.

Then $2\pi rhF = 4\pi \times 2\pi ah\sigma$,

$$\mathbf{F = 4\pi a\sigma/r.}$$

More shortly, $F = 2Q/r$, where $Q$ is the charge per unit length.

If $r$ were less than $a$, a cylinder of radius $r$ would enclose no charge, $\therefore\ F = 0$.

The potential difference between two points outside at distances $r$ and $r_0$ can be shown, by calculus,

$$= -\int_{r_0}^{r} \frac{2Q}{r}\ dr = 2Q \log (r_0/r).$$

The formula does not involve the radius $a$, and therefore holds for a thin rod as well as for a cylinder. A figure (Fig. 26) is drawn to show equipotential lines for a set of potentials in arithmetical progression (cf. Fig. 6).

**45. Two Parallel Cylinders.**—Let $O$ and $O'$ represent their axes. Let their radii be $a$ and $b$, and let $OO' = c$. The cylinders are not supposed to cut one another. If each be outside the other, $c > a + b$, as in Fig. 27. If one be inside the other, $c < a \sim b$ (Fig. 28). The working is the same in both cases; we confine ourselves to the first.

Two points, $A$ and $B$, on $OO'$ can always be found, which are inverse with respect to both cylinders. That is,

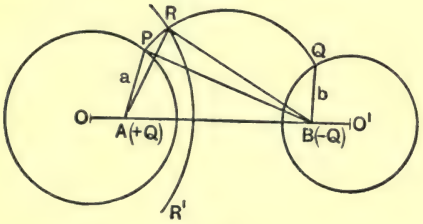



Fig. 27.

$OA \cdot OB = a^2$, and $O'A \cdot O'B = b^2$. Put $OA = x$, then $OB = a^2/x$, $O'A = c - x$, $O'B = c - a^2/x$, and $x$ can be found from the quadratic equation,

$$b^2 = (c - x)(c - a^2/x).$$

$A$ and $B$ can be regarded as rods perpendicular to plane of figure.

Imagine a charge of uniform line-density, $Q$, arranged along $A$, and $-Q$ along $B$. If $P$ be any point on the first cylinder, the potential at $P$

$$= 2Q\{\log BP - \log AP\}$$
$$= 2Q \log (OB/OA) = 2Q \log (a^2/x^2) = 4Q \log a/x.$$

In the same way the potential at $Q$ on second cylinder

$$= 2Q\{\log BQ - \log AQ\} = 2Q \log (O'B/O'A) = 4Q \log b/(c - x).$$

The potentials over each cylinder are therefore constant, and the potential difference is

$$4Q \log \{(ac - ax)/bx\}.$$

Consequently, if two cylinders be kept at this difference of potential, the induced electrifications, $X$ and $Y$, on them will be such that the image of $X$ at all outside points is the distribution $Q$ along the bar $A$; and for all

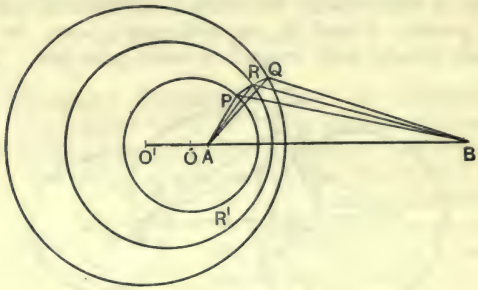



Fig. 28.

inside points is $Q$ along $B$. Also the image of $Y$ is $-Q$ along $B$ for outside points, $-Q$ along $A$ for inside points. For this set of images will satisfy the conditions, namely, that the potential is constant over each cylinder, and that the P.D. is as required between the two cylinders.

**46. Particular Cases.**—If one cylinder be of zero radius (*i.e.* a rod), $b = o$, and we can take $c - a^2/x = o$, $x = a^2/c$. The potential difference is infinite.

If the cylinders be of the same radius, $a = b$. The quadratic reduces to $x^2 - cx + a^2 = o$, giving the smaller root $x = (c - \sqrt{c^2 - 4a^2})/2$. The potential difference is

$$4Q \log (c - x)/x = 4Q \log \frac{c + \sqrt{c^2 - 4a^2}}{c - \sqrt{c^2 - 4a^2}}.$$

If $b$ be infinite, one cylinder becomes a plane. Then $c$ also is infinite; let $c - b = d$ where $d$ is finite. Evidently $c - b =$ the perpendicular distance from $O$ to the plane. The quadratic is

$$(c - d)^2 = (c - x)(c - a^2/x)$$
$$- 2dc + d^2 = - c(x + a^2/x) + a^2.$$

Divide by $c$ and then put $1/c = o$ ;

$$x + a^2/x = 2d,$$

$$x = d - \sqrt{d^2 - a^2}.$$

The potential difference $= 4\,Q \log \frac{a\,(c - x)}{b.x}$

$$= 4Q \log \frac{a}{x} = 4Q \log \frac{a}{d - \sqrt{d^2 - a^2}}$$

$$= 4Q \log \frac{d + \sqrt{d^2 - a^2}}{a}.$$

For $(c - x)/b = 1 + (d - x)/b = 1$ in limit.

**47. Equipotentials and Lines of Force.**—If $R$ be any point in the field of the two cylinders, or of their equivalent rods $A$ and $B$, then the potential at $R$ is $2Q \log \{BR/AR\}$. Hence the equipotentials are such surfaces as have $BR/AR$ constant, that is, they are cylinders of the system coaxial with the original cylinders. The circles, $RR'$, on which such cylinders stand all have a common radical axis and have a common pair of inverse points $A$ and $B$. The lines of force are orthogonal to these coaxials, and therefore are circles through $A$ and $B$, such as $PRQ$.

**Ex. 1.**—Calculate the force with which a unit + ve point charge at $A$ is attracted to an infinite earthed conducting plane at a distance of 2 cm. from the charge. Show that the surface density induced at the point of the plane nearest to the charge is nearly ·04.

**Ex. 2.**—Taking the case of Ex. 1, suppose a unit − ve point charge be placed at $B$, 2 cm. from the plane and 4 cm. from $A$. Show that the resultant force on $A$ will make an angle of 45° with $AB$, and is smaller than before in ratio 1 : ·91.

**Ex. 3.**—If both the charges at $A$ and $B$ in Ex. 2 were unit positive charges, find in magnitude and direction the force exerted on each of them.

**Ex. 4.**—If any number of point charges be before an earthed infinite plane, show that the surface density everywhere is the algebraic sum of the densities due to the point charges separately.

**Ex. 5.**—With the distributions of Ex. 2 and Ex. 3, calculate the density perpendicularly under $A$ and $B$, and the middle point of $AB$.

**Ex. 6.**—$A$ is the centre of an earthed sphere of radius 5 cm. $B$ is a point at distance 7 cm. from $A$. If a unit + point charge be put at $B$, calculate the surface densities at the two points where the straight line $AB$ cuts the sphere. Calculate also the attraction between the sphere and the point charge.

**Ex. 7.**—If, in Ex. 6, a second unit + charge be put at $C$ where $AC = BA$ and is in the same straight line, calculate the resultant force on each point charge.

**Ex. 8.**—If, with figure as in Ex. 7, the charges at $B$ and $C$ are + 1 and − 1, show that the force exerted on either is greater than if the sphere had been absent in the ratio 15·9 : 1.

**Ex. 9.**—If the sphere whose centre is $A$ in Ex. 6-8 had been insulated instead of earthed, but initially uncharged, calculate the difference in the results. Show that Ex. 8 is unaffected.

**Ex. 10.**—A unit + charge is at $B$, 3 cm. from the centre $A$ of a spherical cavity in conducting material of radius 5 cm. Show that it is repelled from the centre of the sphere with a force of ·058 dyne. Find the surface densities at the ends of diameter $AB$.

**Ex. 11.**—Calculate the Mutual Energy in the various cases of Ex. 1-10.

It may be assumed that the mutual energy is minus one half the sum of the products $QV$ formed by multiplying the actual charge $Q$ at each point of the field by the potential $V$ at that point. This agrees with § 54. Each potential is to be taken as due only to the other charges in the field, without counting that at the point considered.

Thus, in Ex. 1, if $A'$ be the image of $A$, the potential at $A$ is that due to the image charge − 1 at $A'$, and is $-1/AA' = -\frac{1}{4}$. Hence the product $-\frac{1}{2}QV$ for $A$ is 1/8. The induced charge − 1 is not actually on the image $A'$, but is spread over the earthed conductor whose potential is zero. Hence $-\frac{1}{2}QV$ is zero for the induced charge, and the energy is 1/8. In Ex. 2 the potentials at $A$ and $B$ are $-(4-\sqrt{2})/8$ and $+(4-\sqrt{2})/8$; and the charges are + 1 and − 1: hence $\Sigma QV = (4-\sqrt{2})/8$. Similarly in other cases.

# CHAPTER III.

## INSULATING MEDIA.

**48. Coulomb's Law in Insulating Media.**—If we use absolute units, we have seen (§ 7) that the force between two charges in air is given by

$$F = q_1 q_2 / r^2.$$

If the same charges be surrounded by turpentine, or any other insulating medium, the law becomes

$$F = q_1 q_2 / k r^2.$$

The constant $k$ is called the **Specific Inductive Capacity,** the **Dielectric Power,** the **Dielectric Constant,** etc. It is greater than 1 in solids and liquids and many gases. For a vacuum, and gases at ordinary pressures, it is so nearly 1 that we can neglect the difference. For turpentine, $k = 2$ nearly; for pure water, $k = 80$.

For many solids $k$ is not constant, but depends on the time for which the solid has been in the electric field.

Using the amended form of Coulomb's law we get, for point charges, when space is filled with a dielectric other than air,

$$F = q/kr^2, \qquad V = q/kr.$$

For a uniformly charged sphere,

| | | |
|---|---|---|
| Internally, | $F = 0,$ | $V = Q/ka.$ |
| Externally, | $F = Q/kr^2$ | $V = Q/kr.$ |

For a disc uniformly charged

$$F = 2\pi\sigma/k.$$

And similarly for other cases.

**49. Lines of Force (Maxwell).**—Let a sphere of radius $r$ be described about a point charge $q$ in an insulating medium. If $r$ be small enough, we can neglect other charges and take the field $= q/kr^2$ anywhere on the sphere.

Let lines of force be drawn as in §§ 14, 15. Choose the number of unitary lines in such a way as to fulfil the condition that the Resolved Field = number of lines per unit area; then the number of lines issuing through the sphere is

$$N = \text{Area} \times \text{Field} = 4\pi r^2 \times q/kr^2 = 4\pi q/k.$$

The lines are Maxwell Lines of Force (§ 17); they indicate the field strength in the most helpful way, but have the disadvantage that the number proceeding from the charge $q$ alters if $q$ be moved from one medium to another.

**50. Lines of Induction and Faraday Lines.**—Next, let us choose our lines in such a way that the number issuing is $4\pi q$. The lines are then called Lines of Induction. They are related to a certain vector called the Induction in precisely the same way as the Maxwell Lines are related to the Force. That is, the resolved induction in any direction = the number of lines of induction per unit area drawn perpendicular to that direction. The Induction due to a point charge $q$ at distance $r$ is $q/r^2$ whatever be the dielectric (supposed homogeneous). Hence the Induction $= k \times$ Field.

Third, let the number issuing be $q$. The lines are now Faraday Lines. They are related to a vector called the Electric Displacement in the same way as lines of induction are to the Induction. Every Faraday line contains $4\pi$ lines of induction, and at every point the Displacement = Induction $\div 4\pi$ = Field $\times\ k/4\pi$.

It is inconvenient having two vectors, Induction and Displacement, related to each other so closely and differing only by a constant multiplier. Faraday Lines are most used in Electrostatics; but in the corresponding work in Magnetism one employs lines of Induction exclusively. Also the constant $k$ is often defined as the ratio

$$\text{Induction} \div \text{Field}.$$

Hence we have to know both definitions. We must even be prepared for confusion in text-books between these quantities.

**51. Boundary Conditions.**—Imagine a point charge $q$, surrounded by a concentric sphere of (say) turpentine of radius $a$, then by air. It must be taken for granted that the field in air at great distances $C$ is independent of the turpentine, and therefore is $q/R^2$; hence there are $q/R^2$ lines of force per unit area of a large sphere of surface $4\pi R^2$; that is $4\pi q$ lines in all. In air $k = 1$, so there is no difference between force and induction. Hence the number of lines of induction in air is $4\pi q$, independent of the turpentine sphere; and they radiate symmetrically

Now $4\pi q$ lines of induction originate from $q$ in the turpentine, and these also radiate symmetrically.

Hence we can infer that the lines of induction, reaching the turpentine-air surface, pass without altering their number from one medium to the other. At an uncharged boundary no lines of Induction stop or start. The same is true of Faraday Tubes.

**52. Refraction of Lines.**—Let $F_1$ and $B_1$ be the field and induction in one medium; then they are in the same direction* and $B_1 = k_1 F_1$. Let them make $\angle \phi_1$ with the normal to a boundary. Let $F_2$, $B_2$, $k_2$, $\phi_2$ refer to the medium $F$ on the other side of the boundary. Since the lines of induction are continuous

the number reaching unit area in the first medium
= the number leaving it in the second, hence

$$B_1 \cos \phi_1 = B_2 \cos \phi_2 \text{ ; or } k_1 F_1 \cos \phi_1 = k_2 F_2 \cos \phi_2.$$

Another relation between the media is derived from the fact that no abrupt change takes place in the potential.† Hence the P.D. alters by the same amount for a small travel $PQ$, where $P$ and $Q$ are consecutive joints on the boundary, whether $P$ and $Q$ are regarded as points in the

* We are considering only isotropic dielectrics.
† Compare, however, § 69 and its footnote.

first medium or second. Hence the field component in direction $PQ$ is the same for both. It follows that the tangential components are equal, or

$$F_1 \sin \phi_1 = F_2 \sin \phi_2.$$

In words: In crossing any uncharged boundary the normal component of the induction and the tangential component of the field are both unaltered. By division $\tan \phi_1/\tan \phi_2 = k_1/k_2$, which is the "law of refraction."

**53. Condensers.**—Mathematically a condenser is an uncharged region occupied by a definite bundle of Faraday Tubes originating from one conducting surface and terminating on another. These conductors are called the Armatures. By the definition, the conductors must have contrary charges each equal to the number of Faraday Tubes. Let these charges be $\pm Q$, and let $V$ be the potential difference (or P.D.) of the armatures, then the **Capacity** $K$ of the condenser is defined as equal to $Q/V$.

Let an ordinary Leyden Jar be charged positively in the usual way, its outer tinfoil being earthed. Nearly all the lines originating from the charged inner tinfoil terminate on the outer tinfoil, and constitute a mathematical condenser according to the above condition. But a few lines usually stray to (say) the walls of the room, which may also be supposed at zero potential. These form, mathematically, a second condenser. The two $+^{ve}$ coatings are metallically connected, and the two $-^{ve}$ coatings are at the same potential (zero), so these condensers are said to be "in parallel."

**54. Energy of a Condenser.**—Let $K$ be its capacity, $\pm Q$ its two charges, $V$ its potential difference, and $W$ its energy. Then $K = Q/V$; $Q = KV$. Let its potential difference undergo a very small increase $\delta V = V' - V$, so that its potential difference becomes $V'$ and its charges become $\pm Q' = \pm KV'$, and its energy becomes $W'$. The increase of energy $\delta W = W' - W$; and the charges are numerically increased by $\delta Q = Q' - Q$.

The alteration is equivalent to transferring the small $+^{ve}$ charge $\delta Q$ from one plate of the condenser to the other. But the P.D. is $V$ and, therefore, ultimately the work done

$$= \delta W = V.\delta Q = KV\delta V.$$

Now

$$V'^2 - V^2 = (V' - V)(V' + V) = \delta V.(V' + V),$$

which ultimately

$$= 2V\delta V;$$

for we can take the change $V' - V$ as small as we please. Hence ultimately

$$\delta W = \tfrac{1}{2}K(V'^2 - V^2)$$
$$= \tfrac{1}{2}K.\delta(V^2).$$

Hence as $V$ gradually increases from zero upward, $W$ increases at the same rate as $\frac{1}{2}KV^2$. And $W = 0$ when $V = 0$; for the condenser is then uncharged. Hence finally

$$W = \tfrac{1}{2}KV^2.$$

Also

$$Q = KV,$$

therefore we have additional equations

$$W = \tfrac{1}{2}QV = Q^2/2K.$$

**55. Simplest Case.**—The simplest case is also the most important. The condenser consists of parallel plates of area $A$ opposed to one another, the distance apart is $d$ and the specific inductive capacity is $k$. We suppose $d$ so small that all the lines from one plate reach the other; the lines of force being parallel and uniformly distributed over the area. The charges are $+ Q$ and $- Q$; the surface densities $\pm \sigma$ where $\sigma = Q/A$ (Fig. 29).

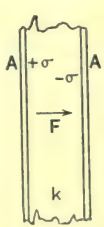


Fig. 29.

The number of lines of induction $= 4\pi Q$,

$$\therefore B = 4\pi Q/A.$$

If the field be $F$, then $V =$ work done when force $F$ acts over distance $d = Fd$;

$$\therefore F = V/d.$$

But

$$B = kF$$
$$\therefore 4\pi Q/A = kV/d$$
$$\therefore K = Q/V \text{ (by def.)} = Ak/4\pi d.$$

*Note.*—The capacity $K$ depends on the medium and varies as $k$; $\therefore$ it $= k \times$ capacity of a similar condenser in which the dielectric was air. This is why $k$ is called Specific Inductive Capacity. We can therefore define the Specific Inductive Capacity as the ratio of the capacity of a condenser in which the plates are separated by the given dielectric to the capacity of a condenser (otherwise identical) in which the plates are separated by air.

**The Energy** of this simple condenser $= \frac{1}{2} QV$.
But

$$B = 4\pi Q/A, \text{ and } F = V/d,$$
$$\therefore \tfrac{1}{2} QV = AdBF/8\pi.$$

But $Ad$ is the volume of the condenser

$$\therefore \text{Energy per unit volume} = BF/8\pi.$$

**56. Field Energy in General Case.**—Consider any electric field whatsoever. In it a series of equipotential surfaces can be drawn; and a series of lines of force cutting them at right angles.

Take an area $A$ on an equipotential so small that it can be regarded as ultimately plane; the lines of force through $A$ determine a corresponding area $A'$ on a consecutive equipotential. The potential difference is supposed so small that $A'$ is ultimately equal and parallel to $A$ and at a small distance from it. The field between these areas is the same as if the two equipotential surfaces were conductors suitably charged, so as to give rise to the same number of lines of force. Hence the energy per unit volume at any point of the electric field is the same as it would be in such a condenser. It therefore

$$= BF/8\pi.$$

**Ex. 1.**—On a very large plane conductor rest successively three layers of dielectric of small thicknesses $d_1$, $d_2$, $d_3$ and specific

inductive capacities $k_1$, $k_2$, $k_3$; and then another plane conductor. If $A$ be area of the opposed conductors and $V$ the difference of potential of the plates, find the charges (Fig. 30).

Let $4\pi Q$ be the number of lines of induction, so that $\pm Q$ are the opposing charges. The induction $B$= number of lines per unit area $= 4\pi Q/A$.

Hence the fields in the three media are successively $4\pi Q/Ak_1$, $4\pi Q/Ak_2$, $4\pi Q/Ak_3$. Then $V$= work done by unit charge travelling distances $a_1$, $a_2$, $a_3$ under these three fields,

$$\therefore\ V = \frac{4\pi Q}{A}\left\{\frac{a_1}{k_1} + \frac{a_1}{k_2} + \frac{a_3}{k_3}\right\},$$

whence $Q$ is found.

**Ex. 2.**—A metal sphere of radius $a$ is in air, and enclosed in a concentric spherical shell of specific inductive capacity $k$ and radii $b$ internal and $c$ external. Outside all is an earthed metal sphere

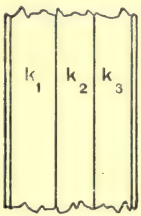



Fig. 30.

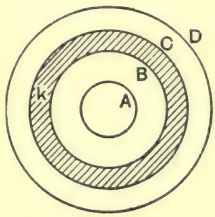



Fig. 31.

of radius $d$ (internal). If $Q$ be the charge given to the inner metal sphere, find the field and potential at every distance $r$ from the centre (Fig. 31).

Let $A$, $B$, $C$, $D$ be potentials at distances $a$, $b$, $c$, $d$. The number of lines of induction radiating from inner sphere $= 4\pi Q$; hence the induction $B$ at distance $r$ = the number of lines per unit area $= 4\pi Q/4\pi r^2 = Q/r^2$.

Where $k = 1$, *i.e.* from $r = a$ to $r = b$; and $r = c$ to $r = d$; $Q/r^2$ = force or field. Hence the work done on unit charge travelling from $r = a$ to $r = b$ is $Q\left(\frac{1}{a} - \frac{1}{b}\right)$, $\therefore\ A - B = Q/a - Q/b$.

Similarly $C - D = Q/c - Q/d$.

Where $k \neq 1$, from $r = b$ to $r = c$, the field $= Q/kr^2$. Hence $B - C = Q/kb - Q/kc$.

The outer sphere is earthed, $\therefore\ D = 0$.

Hence, adding,

$$A = Q\left\{\frac{1}{a} - \frac{1}{b} + \frac{1}{k}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{d}\right\}.$$

If $V$ be the potential at any distance $r$,

$$V = Q\left\{\frac{1}{r} - \frac{1}{d}\right\} \quad \text{if } r \text{ lies between } c \text{ and } d$$

$$= Q\left\{\frac{1}{c} - \frac{1}{d} + \frac{1}{k}\left(\frac{1}{r} - \frac{1}{c}\right)\right\} \quad \text{,, ,, } b \text{ and } c$$

$$= Q\left\{\frac{1}{c} - \frac{1}{d} + \frac{1}{k}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{r} - \frac{1}{b}\right\} \quad \text{,, ,, } a \text{ and } b$$

If there were no external metal sphere, this is equivalent to putting $d = \infty$, $\therefore$

$$A = Q\left\{\frac{1}{a} - \frac{1}{b} + \frac{1}{kb} - \frac{1}{kc} + \frac{1}{c}\right\}.$$

If there were no dielectric shell, this is equivalent to putting $b = c$,

$$\therefore A = Q\left\{\frac{1}{a} - \frac{1}{d}\right\}.$$

**Ex. 3.**—Find the capacity of a condenser formed of two sheets of tinfoil each of area 25 sq. cm., the dielectric being mica of thickness 1/10 mm. and specific inductive capacity 7.

**Ex. 4.**—Find the capacity of a Faraday condenser formed of concentric spheres of radius 10 and 10·5, the dielectric being vaseline ($k = 2{\cdot}2$).

**Ex. 5.**—Assuming the result of § 44, show that the capacity, per unit length, of a condenser formed of two long coaxial cylindrical surfaces, separated by a dielectric, is $k/2\log(b/a)$ ; where $b$ and $a$ are the radii. Calculate the capacity if $k = 2$, length $= 20$, $b = 1{\cdot}1$, $a = 1$.

**Ex. 6.**—If $l$ be the length of a cylindrical condenser, $a$ and $a + x$ be radii, show that if $(x/a)^2$ can be neglected the capacity is $kS/4\pi x$, where $S$ is the mean of the areas of the bounding cylinders.

**Ex. 7.**—Find the capacity, per unit length, of two very long parallel cylinders. (Use § 45.)

**Ex. 8.**—Show the capacity per mile of a telegraph wire of radius 1/8 inch, carried at a uniform height of 20 feet above the earth's surface, is nearly 15,800.

[From § 46 we can deduce the approximate formula, capacity per unit length $= 1/2\log(2h/r)$, where $h$ and $r$ are the height and radius, and the logarithm is to base $e$.]

**57. Joining up of Condensers.**—A set of condensers is *in parallel* if their positive plates be connected and also their negative plates (Fig. 32). Thus all have the same potential difference $V$, and if their capacities are $K_1$, $K_2$, . . . their changes are

$$Q_1 = K_1V, \quad Q_2 = K_2V, \quad Q_3 = K_3V, \ldots$$

If $Q$ be the total charge,

$$Q = Q_1 + Q_2 + \ldots = (K_1 + K_2 + \ldots)V.$$

$\therefore$ The capacity of the combination is

$$K = Q/V = K_1 + K_2 + K_3 + \ldots$$

A set of condensers is *in series* when the negative plate of each is connected to the positive plate of the next (Fig. 33). If they be initially uncharged, and $+ Q$ be given to

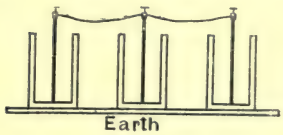


Fig. 32.

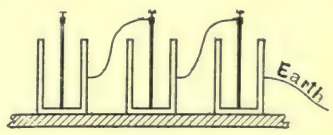


Fig. 33.

$+_{ve}$ plate of first condenser, then $- Q$ appears on its negative plate, and $+ Q$ goes to the $+_{ve}$ plate of the second condenser.

Thus $Q$ is numerically the charge of every condenser of the series, and the successive potential differences are $Q/K_1$, $Q/K_2$, . . .

If $V$ be the total potential difference between the first and last plate, and $K$ the resultant capacity,

then $$V = \frac{Q}{K_1} + \frac{Q}{K_2} + \frac{Q}{K_3} + \ldots = Q(1/K_1 + 1/K_2 + \ldots),$$

$\therefore$ $$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3} + \ldots$$

**Ex. 1.**—Three Leyden Jars of unit capacity are put ($a$) in series and ($b$) in parallel, and charged to potential $V$. In case ($a$), total capacity is $K = 1 + 1 + 1 = 3$; $\therefore$ in ($b$) $1/K = 1 + 1 + 1$, $\therefore$ $K = 1/3$. The energy $= KV^2/2$, $\therefore$ is $3V^2/2$ and $V^2/6$ in the two cases.

If, instead of being charged to same potential, the same charge $Q$ had been given to each system, the $K$'s would be as before, and the energies $= Q^2/2V$ and $\therefore = V^2/6$ and $3V^2/2$ in the two cases.

**Ex. 2.**—Two equal condensers are charged in parallel. They are then disconnected and placed in series. Show that the energy is unaltered, the potential difference doubled, and the available charge halved.

**Ex. 3.**—Two condensers of unequal capacity $a$ and $b$ are charged in parallel to potential $V$. They are then placed in series. The energy is unaltered and potential difference doubled. If the extreme plates are connected, show that the system retains charges $\pm (a - b)V$, and has energy $\frac{1}{2} V^2 (a - b)^2/(a + b)$.

**58. Capacities of Simple Bodies.**—A positively charged conductor in empty space can be regarded as the origin of a set of lines of induction which proceed to zero potential at infinity. In practice, the lines proceed to the walls of the room, which can be treated as an earthed conductor of zero potential. The case of a negatively charged conductor is similar.

Thus a conductor can usually be treated as one of two plates of a condenser; the other plate being at infinity or earthed.

The P.D. of this condenser is the potential $V$ of the conductor. The opposed charges are equal to the charge $Q$ on the conductor. Hence the capacity $K =$ Charge $\div$ Potential $= Q/V$; and the energy $= \frac{1}{2} QV = Q^2/2K = \frac{1}{2} KV^2$.

A sphere, *e.g.*, has capacity $=$ its radius (§ 33). We may add the information that the capacity of a flat circular disc of radius $c$, charged both sides, is $2c/\pi$.

**59. Æpinus Condenser.**—Two flat circular plates of radius $c$ are separated by a narrow air space of width $d$. One is charged, the other earthed. Find the capacity.

The space between has capacity $K_1$ nearly $= A/4\pi d$ (§ 55) if $d$ is not too large. Since $A = \pi c^2$, $\therefore K_1 = c^2/4d$. The back of the charged plate forms with the walls of room a condenser of capacity $K_2$ nearly $c/\pi$. The condensers are in parallel, hence the total capacity $= c^2/4d + c/\pi$ nearly.

**60. Forces acting on an element of Conducting Surface.**—A rectangle of sides $a$ and $b$ is separated by a small distance $d$ from an infinite conducting plate, to which it is parallel. Its charge is $Q$. To find the force exerted on the rectangle perpendicular to the plate, and the force exerted on each edge of the rectangle parallel to the plate (Fig. 34).

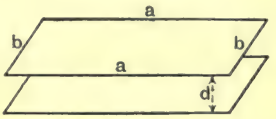


Fig. 34.

The capacity is $K = Ak/4\pi d$ (§ 55) $= abk/4\pi d$. The energy is $E = Q^2/2K = 2\pi d\ Q^2/abk$. If $d$ increases by $\delta d$, $E$ increases by $\delta E$ where

$$\delta E = 2\pi Q^2 \delta d/abk.$$

If $X$ be the force needed to pull the plate outwards, the work done in the displacement is $X\delta d$.

$$\therefore\ X\delta d = \delta E.$$

Hence $$X = \delta E/\delta d = 2\pi Q^2/abk.$$

Since $Q = KV = abk/4\pi d V$, we also have

$$X = kV^2ab/8\pi d^2.$$

The force per unit area $= X/ab = kV^2/8\pi d^2$. Since $V/d = F$, this $= kF^2/8\pi = BF/8\pi$, and is a tension.

Next, if $b$ alters to $b + \delta b$,

$$\delta E = \frac{2\pi dQ^2}{ak}\left\{\frac{1}{b+\delta b} - \frac{1}{b}\right\} = -\frac{2\pi dQ^2\delta b}{ab^2k}$$

ultimately.

If $Y$ be the force acting on the edge $a$, $\delta E = Y\delta b$,

$$Y = -\frac{2\pi dQ^2}{ab^2k}.$$

This force can be regarded as acting on the area $ad$. Hence the force per unit area

$$= Y/ad = -2\pi Q^2/ka^2b^2 = -BF/8\pi$$

as before, and therefore is a pressure.

The forces are therefore the same as if the tension along the lines of force is $BF/8\pi$ per unit area, and the

pressure in every direction perpendicular to the lines of force is also $BF/8\pi$ per unit area.

**Example.**—A plate is midway between two parallel plates $EB$ and $FD$ connected metallically, whose distance apart is $d$. The difference of potential between the plate and the connected plates is $V$. Show that there is a force $V^2/2\pi d$ acting on every centimetre length of the edge of the plate (Fig. 35).

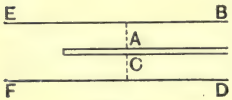


Fig. 35.

The field between the free plate and either connected plate is $V/\frac{1}{2}d$; hence $B = F = 2V/d$. The force per unit area over each space $A$ or $C$ is therefore $BF/8\pi = V^2/2\pi d^2$. But the area of $A$ and $C$ together, per cm. length of the edge, is $d$; hence the force per unit length is $V^2/2\pi d$.

**61. Absolute Electrometer.**—The absolute electrometer measures $V$ by finding the attractive force exerted on an area $A$ by a larger plate charged to a potential difference $V$. This force (by § 60) $= AV^2/8\pi d^2$, the dielectric being air. The force is measured by weights, and $d$ can be adjusted by a screw. Other electrometers of same type estimate the force by springs.

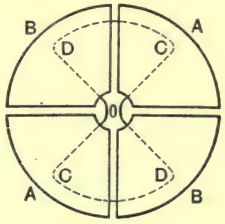
The actual construction of the instruments should be read from descriptive text-books.

**Example.**—The movable plate is a circle of 3 cm. radius. The distance between the plates is always 2 mm., exactly, when a reading is taken. Find the P.D. in absolute units when the attraction is ·1 gm. wt.

The force in dynes $= A V^2/8\pi d^2$, and 1 gm. wt. $= 981$ dynes.
Hence $9\pi . V^2/8\pi \times (\cdot 2)^2 = 98\cdot 1$, $V^2 = 3\cdot 488$, $V = 1\cdot 87$.

**62. Quadrant Electrometer.**—The fixed parts form a conducting (brass) cylindrical box $ABAB$ cut into four quadrants (Fig. 36). Two opposite quadrants $AA$ are electrically connected and therefore at the same potential $V_1$; the other two are connected, and at potential $V_2$. The moving part is a lamina of form shown dotted in figure, suspended at $O$ by a torsion fibre so as to hang and turn horizontally. Let its potential be $V$; and its distance from top or bottom surface be $d$, supposed uniform.

The force per unit length on the edges $OC$, within the quadrants $A$, is $(V - V_1)^2/2\pi d$ as in Ex. to § 60. The sum of moments of these forces about the torsion fibre is a certain multiple of $(V - V_1)^2$; and the sum of moments for the edges $OD$, within the quadrants $B$, is the same multiple of $(V - V_2)^2$. The total moment is




$$C\{(V - V_1)^2 - (V - V_2)^2\}$$
$$= C\{(V - V_1) - (V - V_2)\}$$
$$\{(V - V_1) + (V - V_2)\}$$
$$= C(V_2 - V_1)(2V - V_1 - V_2)$$

where $C$ is nearly, but not quite, constant. If the angle of torsion be $\theta$, this couple $= L\theta$, where $L$ is constant for the suspension, or nearly so.

Fig. 36.

Hence, at least where $\theta$ is not large,

$$(V_2 - V_1)\{V - \tfrac{1}{2}(V_1 + V_2)\} = U\theta$$

where $U$ is a constant for the instrument.

**63. Idiostatic and Heterostatic use of Electrometer.**—The quadrant electrometer is used *heterostatically* if the three potentials $V$, $V_1$, $V_2$, are all different. Usually $V$ is large and constant, and $V_1$, $V_2$, small.

$$\theta \propto V_1 - V_2, \text{ nearly; or } \theta = \lambda(V_1 - V_2),$$

and one compares potential differences by the angular deflection produced. A standard cell or battery is used, giving a known $V_1 - V_2$; then the observed $\theta$ gives us $\lambda = \theta/(V_1 - V_2)$. The quantity $\lambda$ is then regarded as constant for a series of experiments, until it is thought necessary to test its value again by the standard cell.

The electrometer is used *idiostatically* if the moving part or "needle" be connected to a pair of quadrants $B$. Then $V = V_2$, hence

$$\theta \propto (V - V_1)^2,$$

so that the angle varies as the square of the P.D.

**Ex. 1.**—It is found that a quadrant electrometer turned 20 divisions when idiostatically connected to a P.D. of 50 volts.* When used heterostatically its quadrants were connected to 1·5 volts and the turn was 15 divisions. What was the potential of the needle?

Evidently if $V$ be the potential required

$$20U = \tfrac{1}{2}\,50^2,\ 15U = V.\,1{\cdot}5,$$

whence $$V = 625.$$

**Ex. 2.**—Show that if the space between the plates of an absolute electrometer, or within the quadrants of a quadrant electrometer, were occupied by turpentine ($k = 2$), the force would be double for the same potential difference.

**Ex. 3.**—It is observed that a quadrant electrometer deflects 20 divisions for 1·44 volts (or 1·44/300 absolute units of potential). If the quadrants be connected to the plates of a standard condenser of capacity 1,000 cm., find what charge given to the condenser is indicated by a deflection of one division. The instrument is used heterostatically.

**Ex. 4.**—The deflection is 14 when the quadrants are connected to the charged standard condenser of capacity 1,000. It falls to 11·5 when a Leyden jar is connected in parallel with the condenser. Find the capacity of the jar.

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* For the definition of the volt see later. It can be regarded as 1/300 absolute units of Potential.

# CHAPTER IV.

## MISCELLANEOUS ELECTROSTATIC PHENOMENA.

**64. Electroscopes.**—If two gilt pith balls are suspended side by side by equal parallel cords and then charged, they will repel each other to a distance roughly proportional to the (charge) $^{2/3}$ (see § 7, Ex. 2). If the balls are charged by contact with a conductor they are charged to the potential $V$ of the conductor, and if $K$ is the capacity of the balls, the charge imparted is $KV$. Since $K$ is approximately equal to the sum of the radii of the balls, it is constant; and therefore the deflection of the balls $\propto V^{2/3}$. Thus the pith ball electroscope really measures Potential.

In the Gold Leaf Electroscope there are two equal gold leaves hanging close together from a metal support which connects them to a brass ball or disc. The support is well insulated, and the leaves are protected by a glass bottle from draughts, etc. Here again the deflection $\propto V^{2/3}$, where $V$ is the common potential of the leaves, *i.e.* the potential to which the disc is raised. Common instruments should make a potential of one electrostatic unit (*i.e.* 300 volts, see § 103) quite obvious, and detect one tenth of this.

**65. Condensing Electroscope.**—Let a thin plate of mica be laid on the disc of the electroscope and a brass plate (provided with an insulating handle) laid on this. The combination forms a Condensing Electroscope. The plates and the mica constitute a condenser of large capacity $K'$. If its plates be connected to the two terminals of an accumulator, they will be charged to its very small P.D.:

call it $v$. The charge on each plate is $K'v$. When the terminals and then the top plate are removed, the disc and leaves retain the charge $K'v$, but their capacity is now the ordinary capacity $K$ of the instrument. Hence the potential is $V = K'v/K$, and it should be quite large enough to be detected. We are led to infer that accumulators and galvanic batteries generally have their terminals at different potentials. These differences are commercially measured in Volts, where 300 volts = 1 electrostatic unit.

An electroscope situated in infinite space would measure P.D. above or below the absolute zero of potential, *i.e.* the potential at an infinite distance from all attracting bodies. An electroscope situated within a hollow conductor measures the P.D. above or below the potential of the conductor. A room in a laboratory can be regarded as an earthed conductor, so electroscopes measure potential referred to the earth as zero. But the room is not an ideal earthed conductor if electrical experiments are proceeding therein; hence when real accuracy is wanted one should use an electroscope with a metal case, or a surrounding wire cage, connected to a good earth contact such as a water pipe.

**Ex. 1.**—A hollow conductor* stands on the disc of an electroscope. A proof ball (*i.e.* a metal ball insulated by being carried on the end of a vulcanite rod) is charged to potential $V$, and then held within the conductor, and ultimately touches it. Describe what happens.

Assume the ball charged +vely. As it approaches it raises the potential of the system consisting of the hollow conductor, and the gold leaves. When it is completely within the conductor, the potential of the latter ceases to alter. Thus the gold leaves diverge gradually, reaching a maximum when the ball is inside. Their divergence measures the repelled induced charge, which is equal to the charge on the ball, and of the same sign. The contact completely discharges the ball and neutralises the attracted induced charge; it therefore does not alter the deflection at all. The total effect is that the original charge on the ball is transferred entirely to the conductor and electroscope.

**Ex. 2.**—The experiment commences as before, Ex. 1. But when the ball is well inside, the pail is earthed for an instant, then the ball is withdrawn. What happens?

The leaves diverge to a maximum as before. The earthing reduces the potential of the pail and leaves to zero, and the leaves collapse. The repelled induced charge is removed, and the attracted induced

* For shortness we call it a Pail. Ice pails were used by Faraday.

charge completely masks the effect of the ball at all external points. When the ball is withdrawn, this attracted charge goes to the outside of the pail and spreads to the leaves. The charge on the pail and electroscope is equal and contrary to the original charge on the ball; and the divergence now ought to be equal to the orginal divergence.

**Ex. 3.**—The ball is charged to potential $V$, and its charge is given completely to the instrument, as in Ex. 1, producing a divergence of 10 scale divisions. It is again charged to the same potential $V$ and is again inserted and made to touch the inside of the pail. What divergence would one expect? How many such contacts would produce a divergence of 40 divisions?

However great a charge the pail may have, this resides wholly on its outer surface; and does not affect the process of giving an additional charge equal to that of the ball. Consequently two contacts give double the charge.

Now the deflection (if not too large) $\propto$ (charge)$^{2/3}$. If therefore $\theta$ be the deflection when the charge is doubled,

$$\theta/10 = 2^{2/3}, \quad \theta = 15{\cdot}9.$$

If $n$ contacts give a deflection 40, $40/10 = n^{2/3}$, $n = 4^{3/2} = 8$.

**Ex. 4.**—The deflection of a Galvanometer is 48 units. It is left for an hour and then, owing to leakage across the insulation, its deflection is found to be 36. What will the deflection be in another hour?

When a conductor leaks to earth, the rate of leakage is proportional to the P.D. The mathematical theory of such leakage is given in § 189; and it is there shown that the logarithm of $V$ diminishes at a constant rate. Now $V \propto \theta^{3/2}$, hence $\log V = \text{constant} + \frac{3}{2}\log\theta$. So if $\log V$ diminishes uniformly, $\log\theta$ does so also. If the final deflection be $\theta$,

$$\log 48 - \log 36 = \log 36 - \log\theta,$$
$$\tfrac{48}{36} = 36/\theta\,; \quad \therefore\ \theta = 27.$$

**Ex. 5.**—An Æpinus Condenser (§ 55, the dielectric being air) has one plate earthed and the other connected to an electroscope. If the plates are moved, find the theoretical connection between $d$ and $\theta$ when $d$ is small.

The charge is constant and the capacity $\propto 1/d$, hence $V \propto d$. But $V \propto \theta^{3/2}$, hence $d^2/\theta^3$ is constant.

**Ex. 6.**—The space between the plates of the above condenser is filled with a liquid whose specific inductive capacity is $k = 2$. How does $\theta$ alter?

Capacity $\propto k$, $\therefore\ V \propto 1/k$. But $V \propto \theta^{3/2}$ $\therefore\ \theta \propto k^{-2/3}$. In this case $\theta$ is multiplied by $2^{-2/3}$ or ·63.

**Ex. 7.**—The deflection is 40 when the gold leaves are connected to a charged Leyden Jar. The jar is made to share its charge with an identical jar; which is afterwards discharged. The operation is repeated as often as desired. Give an estimate of the deflections after successive operations.

Each operation halves the potential and multiplies the deflection by $2^{-2/3}$ or ·63 nearly. The deflections are roughly 40, 25, 16, 10, 6·3, 4, 2·5, . . . .

**66. Surface Films.**—Researches on Capillarity have established the fact that when two different media are in contact there is no absolutely abrupt transition, across a bounding surface of no appreciable thickness, from the properties of one to those of the other. There is always an intermediate region, called a Surface Film, within which the properties alter by degrees. The simplest case is where water at 15° (say) is in contact with its saturated vapour, no other substance being present. Assuming the boundary horizontal, we have a plane $AA'$ (Fig. 37) below which there is ordinary water, all in the same state, of density nearly $= 1$. We have also a plane $BB'$ above which there is only water vapour, whose density (for saturation at 15°) is nearly $=$ ·000013. Between these we have water matter of densities taking all values from 1 to ·000013 as we pass from $AA'$ to $BB'$. The thickness of the film, from $AA'$ to $BB'$, has been investigated. No great precision is possible, but the distance is of the order of $10^{-6}$ cm.

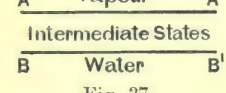

Fig. 37.

A more complicated case is where two liquids meet; say water rests on mercury. Here again we have a surface film. Taking the same figure as before, we could say there is pure mercury below $AA'$, pure water above $BB'$, and the mercury density alters between $AA'$ and $BB'$ from its full value to zero; the water density altering at the same time from zero to 1.

**67. Intrinsic Pressure.**—In the case, above considered, of water and vapour, the pressure above $BB^1$ in

the vapour is quite small. In the liquid it is enormous; the largest estimate (Dupré) being 25000 atmospheres. The difference is called the Intrinsic pressure. It is instructive to consider why this large quantity is not readily detected by experiment.

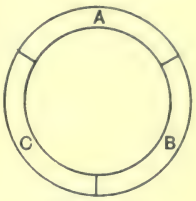

Fig. 38.

Let $A$, $B$, $C$ be three liquids in a re-entrant glass tube (Fig. 38) whose axis is horizontal, so that we may neglect pressure effects due to gravity. When equilibrium is attained, the pressure throughout each liquid is constant; call them $P_A$, $P_B$, $P_C$. Also, where $A$ and $B$ meet, the difference of pressure at opposite sides of the surface film is the contact pressure difference of $A$ and $B$; call it $I_{AB}$. Hence

$$I_{AB} = P_A - P_B .$$

Similarly $$I_{BC} = P_B - P_C , \; I_{CA} = P_C - P_A ;$$

$$\therefore \; I_{AB} + I_{BC} + I_{CA} = 0.$$

This is a relation which holds between the intrinsic pressures for any three substances capable of resting together in equilibrium. If $C$ be vacuum, we can put $I_{AB} + I_B - I_A = 0$, so that (*e.g.*) The intrinsic pressure difference of water and mercury in contact = the difference of intrinsic pressures of water and of mercury. This relation makes it impossible to detect intrinsic pressure by a mercury pressure gauge.

**68. Contact Electrification.**—The above seems to have no direct bearing on Electricity. It is inserted because it is closely analogous to another phenomenon quite different.

When two substances are in contact, the electrical potential is different at the two sides of the surface film $AA'$ and $BB'$ (Fig. 37), and has continuously varying values for intermediate points.

We cannot now discuss the cause of the difference. Let $A, B, C$ (Fig. 38) be three conductors, *e.g.* metallic wires, joined in a re-entrant series. If there be equilibrium, the potential in each one is constant; call them $P_A$, $P_B$, $P_C$. And the potential difference across the surface films may be denoted by $I_{AB}$, etc., where

$$I_{AB} = P_A - P_B, \quad I_{BC} = P_B - P_C, \quad I_{CA} = P_C - P_A.$$

Hence $$I_{AB} + I_{BC} + I_{CA} = 0.$$

Note that this equation is not proved for three bodies in equilibrium if one of them, say $C$, is an insulator. For an insulator can be in equilibrium when the potential within it is not constant. The equation holds for three metallic conductors at the same temperature. For many non-metallic conducting circuits, or metals, if the temperature be not uniform, electric equilibrium does not exist, and the equation fails.

**69. Apparent and Real Contact Potential Difference.**—Let $I_A$ be the intrinsic potential difference between a metal $A$ and the air just in contact with it, but separated from it by the surface film. If $P_A$ be the potential of the metal, and $V_A$ of the air, then $P_A = V_A + I_A$.

Similarly for another metal $B$, $P_B = V_B + I_B$.

In § 28 we provisionally regarded the potential of a conductor as that of air in contact with it.* So, with this convention, the potentials of $A$ and $B$ would be $V_A$ and $V_B$; and the potential difference is $V_{AB} = V_A - V_B$. The true potential difference is, of course, $P_{AB} = P_A - B_B$.

Hence $V_{AB} = P_{AB} + (V_A - P_A) - (V_B - P_B) = P_{AB} - I_A + I_B$.

Similarly for other substances $B$ and $C$,

$$V_{BC} = P_{BC} - I_B + I_C, \; V_{CA} = P_{CA} - I_C + I_A.$$

Adding, $V_{BC} + V_{CA} + V_{AB} = P_{BC} + P_{CA} + P_{AB}$.

For metals both sides of the equation are zero.

* In § 52 we regarded the potential of an insulator in the same way. The result there deduced would be unaffected by a constant intrinsic potential difference.

The distinction between $P_A$ and $V_A$ has been overlooked until quite recently. Volta supposed the potential difference of zinc and copper to be fairly large, nearly a volt; and Ayrton and Perry by careful experiments established it as ·73 volt. Their researches were really on the apparent Potential Differences, $V_{AB}$. It is generally supposed that $P_{AB}$ for copper and zinc is about $10^{-6}$ volts. This is calculated from thermal effects (as in § 241). If the validity of the method be not admitted, it becomes impossible to fix any one contact P.D. with certainty.

A few of Ayrton and Perry's results are given below.

The contact P.D.'s are given for carbon and certain metals. The contact P.D. of two metals is the difference of their P.D.'s with carbon. Thus the P.D. of Zn and Cu is 1·096 − ·370 = ·726. Carbon is electropositive to the metals named.

| | | | |
|---|---|---|---|
| Copper ... ... | ·370 volt | Zinc ... ... | 1·096 volts |
| Iron ... ... | ·485 ,, | Amalgamated Zinc | 1·208 ,, |
| Lead ... ... | ·858 ,, | Brass ... ... | ·414 ,, |
| Platinum ... | ·113 ,, | | |
| Tin ... ... | ·795 ,, | | |

These are really P.D.'s of the air films in contact with the connected metals. Different results might be expected in another gas.

**70. Contact Potential Difference of Insulators: Electrification by Friction.**—Let the substances be $A$ and $B$. The intermediate film can be regarded as a conductor; but an exceedingly bad one. Under the influence of forces not yet thoroughly understood, positive electricity is urged one way through the film, to the $A$ boundary say, and negative to the $B$ boundary. The film conducts badly, therefore the process is not instantaneous. When it is complete, we have layers of electrification of surface density $+\sigma$ on $A$, $-\sigma$ on $B$; the potential difference $V_{AB}$ having its normal value necessary for equilibrium. The surface film is so narrow that it forms a condenser of considerable capacity per square cm., hence the charges on its opposed faces are often very large although the P.D. is quite small, say of the order of a volt.

If the substances be drawn rapidly apart, then the film

is usually so bad a conductor that both sides of it retain their charges.

But these charges are no longer on the plates of a condenser of high capacity, hence the separated insulators now have a very large P.D. This explains what is miscalled Electrification by Friction; for the rubbing is only to ensure close contact over all parts of the surface successively. Conductors are not capable of being raised to so high a P.D. on separation. As the bodies are drawn apart, the separated charges flow to the parts still in contact, and when separation is completed the charges are only as great as if the contact had been over a minute area. The capacity of the opposed surfaces is very small as separation ends.

**71. Electrification by Splashing.**—Pure water and air have a contact P.D., the water being positive. If a water drop falls and splashes, the film is first extended and then torn apart. The water spray is positively electrified, the air around is negative. Negative potential is observable* around waterfalls, but $+^{ve}$ in the actual spray. Curiously, salt water has opposite properties: the air near the sea is $+^{ve}$ and the spray $-^{ve}$.

**72. Electrophorus.**—Typically this consists of two parts. One is an ebonite disc $D$ mounted on a brass plate $S$ called the sole; the other is a circular brass plate $P$, furnished with a knob, and mounted on an insulating handle $H$ of varnished glass.

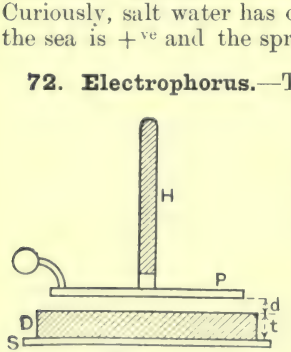

Fig. 39.

The sole is to be always kept at zero potential by being metallically connected to earth. The disc is electrified negatively to a high potential by friction with catskin. The plate is then laid on it. There is true contact at very few points; effectively a narrow air space lies between $D$ and $P$.

* By method explained later, § 81.

Let $A$ be the area, and $a$ the radius, of $P$. Let $d$ and $t$ be thicknesses of air space and vulcanite. Let $-Q$ (negative) be the charge on the disc.

A positive and negative charge, $+q$ and $-q$, will be induced on lower and upper forces of the disc. The plate is now earthed for an instant: enough positive electricity will flow from earth to neutralise $-q$ and raise the positive charge on $P$ to the larger value $q'$. $P$ is now lifted by the handle $H$, used in experiments which require a positively charged conductor.

**73. Charges on Electrophorus before Earthing.**—We have to consider three electric fields (or condensers), traversed by different sets of Faraday tubes. Neglecting that part of $D$ which is not covered by $P$, there is—

First, the condenser formed between the rubbed surface of the vulcanite and the sole. Its capacity $K_1 = Ak/4\pi t$, if $k$ be the specific inductive capacity of ebonite. But $A = \pi a^2$, therefore $K_1 = ka^2/4t$.

Second, between the rubbed surface and the plate. Its capacity $K_2 = A/4\pi d = a^2/4d$.

Third, between the top of $D$ and the walls of the room. Its capacity $K_3$ can be taken as half that of a disc in empty space, and therefore as $a/\pi$. It is evident that $K_3$ is very small, and usually $K_2$ is much larger than $K_1$.

If $-Q$ be the charge on the area $A$ of disc, there are altogether $Q$ Faraday tubes in condensers $K_1$ and $K_2$; say $q$ from the plate and $Q - q$ from the table. Hence $K_1$ has a potential difference $V = (Q - q)/K_1$. The sole is earthed, therefore the top of the vulcanite is at potential $-V$.

Since $q$ is the $+^{ve}$ charge on the under side of the plate, $-q$ is the negative charge on the upper side, and $+q$ is the charge induced thereby on the walls of the room. Hence the condensers $K_2$ and $K_3$ each has $q$ Faraday tubes; and their potential differences are $q/K_2$, $q/K_3$. But the walls of the room can be assumed at zero potential, therefore $-q/K_3$ is the potential of $P$, and $-(q/K_3 + q/K_2)$ is that of the rubbed surface.

Hence $$\frac{q}{K_3} + \frac{q}{K_2} = \frac{Q-q}{K_1}, \therefore \frac{q}{Q} = \frac{1}{K_1} \Big/ \left\{ \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3} \right\}$$

**74. Earthing the Plate.**—This destroys all the Faraday tubes in the field $K_3$, and practically makes $1/K_3 = o$. The fields in $K_1$ and $K_2$ can be called $Q - q'$ and $q'$. The potential differences are equal, say $V'$, therefore

$$V' = (Q - q')/K_1 = q'/K_2,$$

$$\frac{q'}{Q} = \frac{1}{K_1} \Big/ \left(\frac{1}{K_1} + \frac{1}{K_2}\right),$$

which proves, of course, that $q' > q$.

As approximations, $K_3$ in the expression for $q$ (last paragraph) is very small indeed, so $1/K_1$ and $1/K_2$ are negligible in comparison with $1/K_3$.

Hence $$q/Q = K_3/K_1,$$

and $$\therefore\ q'/q = \frac{1}{K_3} \Big/ \left(\frac{1}{K_1} + \frac{1}{K_2}\right).$$

**Ex. 1.**—If the plate has a radius of 10 cm. and thickness 1 cm., and the air film be ·05 cm., find $K_1$, $K_2$, $K_3$. The specific inductive capacity of ebonite may be taken as 3. Find also $q$ and $q'$ in terms of $Q$.

Here $$K_1 = ka^2/4t = 75\,;\ K_2 = a^2/4d = 500\,;$$

and $$K^3 = a/\pi = 3 \text{ nearly.}$$

Hence $$\frac{q}{Q} = \frac{1}{75} \Big/ \left\{ \frac{1}{75} + \frac{1}{500} + \frac{1}{3} \right\} = \frac{20}{523} = \frac{1}{26} \text{ nearly.}$$

And $$\frac{q'}{Q} = \frac{500}{575} = \frac{20}{23}.$$

**Ex. 2.**—The plate of an electrophorus is not laid on the ebonite, but held above it at a distance equal to the thickness of the disc. Find $q'/Q$.

Here $K_1$ and $K_2$ are condensers of the same dimensions, hence $K_1/K_2 = k/1$. Hence $q'/Q = K_2/(K_1 + K_2) = 1/(1 + k)$. If $k = 3$, $q' = Q/4$.

**75. Sparks.**—Two large parallel metal plates,* just convex enough to localise the electric discharge, are kept at a constant P.D. and gradually approached. At a certain distance, $x$ centimetres, a spark passes. The

* The sparking distance is less for spherical knobs of small radius.

relation of $V$ and $x$ (in electrostatic units, implied in § 10) deduced by Chrystal from Leibig's experiments is $V = 4{\cdot}9997 + 99{\cdot}593x$ in dry air, which gives, in round numbers, $V = 5 + 100x$.

The field $F = V/x = 100 + 5/x$.

The surface density (§ 29) is $F/4\pi =$ nearly 8, when $x$ is not very small. When one spark has passed a second passes much more readily, and sparking continues if the plates are drawn further apart. Each spark makes a track in the air of much diminished resistance.

An **Arc** is a region rendered so conducting by the previous passage of electricity that it continues to allow the electric current to pass. The arc is formed by bringing its terminals so near that sparking begins, then, when the interspace becomes a conductor, drawing them apart gradually. Here the conductivity is mainly due to high temperature, for the atoms of a gas which has been dissociated by intense heat can carry electric charges. But the conductivity caused by small intermittent sparks is due to carriers whose nature is not well known but which are more complex than the normal gaseous molecule.

**76. Electric Wind.**—If the density on a conductor be greater than about 8 units, then, even if no conductor be near enough, it seems possible that a minute spark would travel to any minute carrier that was suitably near. The spark would manufacture other carriers. Thus carriers would be continuously produced, and then charged like the conductor and repelled. They would not attain a great velocity, being hampered by frequent collisions with gaseous molecules; but they would eventually set the air in motion and produce a perceptible current.

This is the most likely explanation of the electric wind originating from projecting points on a conductor, and of the luminous brush discharge. We regard the mathematical theory of discharges as quite outside the scope of this book, but those who desire information should study J. J. Thomson, *Conduction of Electricity through Gases*. We content ourselves with the approximate statements (which in exceptional circumstances are very inaccurate

indeed) that the sparking distance is about 1 cm. per 100 electrostatic units of P.D. (or 30,000 volts, § 65) and that a brush discharge takes place if the density exceeds about 8 units per square cm.

**Ex. 1.**—A small Leyden Jar of capacity 10 is being charged gradually by any kind of electric machine. Its knob is ½ cm. from the knob of a large jar. Show that the charge given to the large jar is nearly proportional to the number of sparks that pass.

Let $k$ be capacities of small and large jars, and $V = P.D.$ needed for a spark. When $V$ is reached, the spark formed is (for the instant) so good a conductor that both jars are brought to same potential. If charge $q$ passes, the potential of the large jar rises by $q/K$ and that of the small one falls by $q/k$, hence

$$q/k + q/K = V.$$

Hence the charge $q$ given by each spark is constant. If $K$ is very much larger than $k$, we have nearly

$$q = kV.$$

As an approximation we have $V = 5 + 100 \times \cdot 5 = 55$, and $k = 10$, therefore $q = 550$. This is the principle of Lane's Unit Jar.

**Ex. 2.**—What is the P.D. of the terminals of a Ruhmkorff coil giving an inch spark?

1 inch = 2·54 c.m. By formula, $V = 5 + 100 \times 2{\cdot}54 = 260$ nearly. Changing to volts we get 78,000. But since a much smaller P.D. will maintain a series of sparks than will create a new spark, and since the sparking terminals are not flat plates, the actual P.D. is less. The usual estimate is about 40,000 volts per inch.

**Ex. 3.**—A small jar, gradually charged by a machine, sparks at intervals to a large jar whose knob is at a distance ½ cm. Assuming $V = 5 + 100\,x$, find the capacity of the small jar so that every spark may carry 1,000 units of charge.

**77. Properties of Sparks.**—A spark is an exceedingly narrow column of incandescent gas. Although only a small proportion of the energy of discharge of a Leyden Jar appears in the spark, the local heating is enormous. The air traversed is largely ionised (see Chap. V.), and becomes temporarily a good conductor. Its volume increases with such rapidity as to produce an explosive increase of pressure; this produces the shattering effect when lightning strikes a tree or building, and further damage may

ensue from the heating effects which fuse metals and set wood, etc., on fire.

Because the air becomes a conductor, another spark is able to follow the same track with great ease. Ordinarily, what we call a spark is a series of sparks, of an oscillatory nature, leaping back and forward until equilibrium is effected.

If sparks be passed through water, they often produce decomposition into hydrogen and oxygen. This is mainly due to the enormous rise of temperature. Sparks passed through the oxides of metal can reduce them, restoring the pure metal.

On the other hand, a spark passed through the mixed gases hydrogen and oxygen causes them to combine explosively into water. This is the principle of the Eudiometer. Other reactions are produced by sparks, *e.g.* the conversion of oxygen $O_2$ into ozone $O_3$.

**Example.**—What is the total energy released when a 1 cm. spark passes between terminals of a jar of capacity 50?

If the spark is 1 cm. long, the P.D. is about 105. The charge $= 50 \times 105$, hence the energy is $\frac{1}{2}.50 \times (105)^2 = 280{,}000$ ergs.

**78. Projecting Points.**—If two distant spheres be connected by a fine wire and gradually charged, their potentials $V$ will always be the same, and therefore their charges will be in the ratio of their capacities, *i.e.* their radii (§ 58). If $a$ and $b$ be radii, the charges are $Va$ and $Vb$, and the surface areas are $4\pi a^2$ and $4\pi b^2$, hence the surface densities are $V/4\pi a$, $V\,4\pi b$. Obviously the density is greater on the smaller sphere, and a convection discharge takes place from it first.

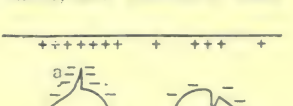

Fig. 40.

A projecting sharp point is comparable to a sphere of very small radius. The density on such a point is enormously greater than on the rest of the conductor, and therefore the electrification on the part of the conductor immediately round the point is rapidly convected off.

Insulating plates cannot easily be charged by contact with a conductor, but are readily charged by the stream of electricity from a projecting point on the conductor. Points on conductors appear also to collect electricity. Imagine a $+^{vely}$ charged surface, conducting or insulating, near a point projecting from an insulated conductor. The $+^{ve}$ charge induces $-^{ve}$ electricity round and on the point. The density on the point is so great that a convective negative discharge pours from it, and gradually neutralises the $+^{ve}$ on the surface, leaving $+^{ve}$ on the conductor. This continues till the conductor and the surface are very nearly at the same potential.

These properties are possessed only by points projecting outwards, as at *a*. Such points as *b* and *c* are useless.

**79. Electrical Machines : Inductive.**—The student should compare the simplified explanation below with a Wimshurst machine.

The twelve lines, *a*, *b*, *c*, . . . . *l*, represent any large

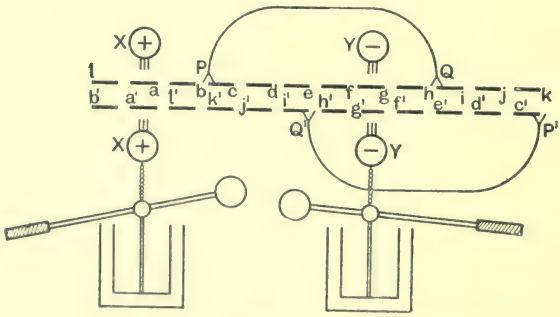


Fig. 41.

number of tinfoil conductors following each other consecutively. These in practice are pasted on a circular rotating disc of varnished glass, vulcanite, or celluloid, so that *l* in figure is next to *k*, and there is no break in the sequence. In the same way *a'*, *b'*, . . . . *l'*, are tinfoils on

a second disc, close to the first, and rotating at the same rate in the opposite direction. $XX$ is a conductor on insulating supports which has sharp points directed inwards, as in Fig. 41; and $YY$ is a similar one diametrically opposite *to* it. Each of these is connected to a Leyden Jar to give it larger capacity, and to an adjustable metal bar with a knob at the end. $PQ$ is an insulated metal wire with metal brushes at its ends; similarly for $P'Q'$.

Supposing the machine actually at work; $X$ and $Y$ having positive and negative charges respectively as indicated. The region around $P$ is nearer $X$ than $Y$, and is therefore at a higher potential than the average, while that around $Q$ is lower. $PQ$ itself, which is in momentary contact with $c$ and $i$, will therefore have induced $-$ and $+$ charges around $P$ and $Q$; so that $c$ is charged $-$ and $i$ is $+$. Similarly $d$ and $e$, which have just passed $P$, are $-^{ve}$, and $j$ and $k$ are $+^{ve}$. Now $e$ lowers the potential around $Q'$, while $i$ is raising that around P′. This effect of the carriers is of the same nature as that due to $X$ and $Y$; for P′ is nearer to $X$ and $Q'$ to $Y$. Hence a positive charge is induced on the conductor passing $Q'$, and a negative on that at $P'$. It follows that $c'$ is $-^{ve}$ and $i'$ is $+^{ve}$; and in the same way $d'$, $e'$ have recently been made $-^{ve}$ and $j'$, $k'$ $+^{ve}$.

Thus the machine has a tendency to continue going when once started. The stream of $-^{vely}$ charged tinfoils leaving $P$ and $P'$ and acting inductively when they get opposite $Q'$ and $Q$ ultimately reaches $Y$; where the projecting points discharge them with almost the same completeness as if there had been actual metallic contact. They are practically within a hollow conductor and touching it. Similarly the $+^{vely}$ charged tinfoils from $Q$ and $Q'$ are increasing the $+$ charge on $X$.

When $X$ and $Y$ are charged to a high enough P.D., a spark passes between the knobs. But even if their discharge is complete (*e.g.* if the knobs are placed close together), electricity is still generated by the inductive action of the carriers themselves.

The machine will therefore work if once started. Usually one of the metal brushes $Q$ is made so much

longer than the other three that its "friction" produces enough $-^{ve}$ electrification on the disc around $Q$ as to induce a little $-$ and $+$ at $P$ and $Q$, and so charge $c$ and $i$ slightly at start. These original small charges are increased by the cumulative inductive action.

**Limit of Potential Difference Attainable.**—If $V$ be P.D. of $X$ and $Y$, $\frac{1}{2}V$ is, approximately, the P.D. of $X$ and $PQ$ or $P'Q'$. The nearest distance from $X$ to either of these wires is considerably less than half the radius $r$ of either disc. Say that it is $r/4$. Then $\frac{1}{2}V$ cannot exceed the sparking potential for $r/4$ cm.; therefore the greatest value of $V$ is about $50r$ electrostatic units. The longest spark between the knobs is not likely to exceed a quarter of the diameter.

**80. Electrical Machines: Frictional.**—In frictional electrical machines a disc of glass is rotated against a rubber of leather smeared with amalgam of mercury and tin. The glass is $+^{vely}$ electrified, and the electricity is collected from it by a prime conductor, similar to $X$ above, armed with points and connected to a Leyden Jar. The $-^{ve}$ electricity on the rubber is usually allowed to escape to earth. A limit exists to the possible P.D. of the prime conductor and rubber. When this is approached sparks begin to pass along the surface of the glass disc, which seems to afford an easier channel than the air. Quite generally, the P.D. of any machine is limited ultimately by the size of the machine, but often a machine can have its P.D. increased by care bestowed on effective insulation.

When electrification is developed frictionally, the work done in friction is largely dissipated in heat. The energy of the charged surfaces is almost wholly given by overcoming the electrical attraction between glass and rubber, charged by contact. Hence it is equivalent, nearly, to the mechanical work done in drawing the charged surfaces apart.

When electrification is developed inductively, as in the Electrophorus and the Wimshurst, the source of the electrical energy is again the mechanical work done in drawing opposite charges apart; *i.e.* the plate from the

disc, in the Electrophorus, and the carriers from those conductors which induced these charges. If all waste of energy, through mechanical friction, could be eliminated, and if a shower of sparks is produced so that $C$ units of electricity (electrostatic) pass per second with a mean potential difference, $V$, the mechanical work supplied per second will be $CV$ ergs.

**81. The Water Dropper.**—A small metal water vessel has a projecting tube from which the water falls in drops. It is connected to an electrometer, and is placed at a point around which the prevailing potential of the air is to be found.

Let $V$ be the potential of the apparatus, $U$ the prevailing air potential, $P$ the air potential just round the tube, as modified by the presence of the apparatus. Then $P$ must lie between $U$ and $V$, and nearly $= V$. If we suppose $U$-$V$ positive, then $P$-$V$ is small and positive.

A drop of potential $V$ growing in a region of higher potential $P$ must have a negative induced charge. When it falls off, it carries away negative electricity, and therefore increases the potential $V$ of the apparatus. In this way $V$ is made nearer to $U$. It can be seen that $V$ would also be made nearer $U$ if $U$-$V$ were negative.

A rapid succession of drops, each of which brings $V$ nearer $U$, finally makes $U$-$V$ negligible. Then the electrometer registers $U$, the potential of the undisturbed air.

# CHAPTER V.

## FLOW OF ELECTRICITY IN ELECTROLYTES.

**82. Electromagnetic Fundamental Units.** — For reasons that will appear later other units of charge and potential are used than those defined in §§ 5-7. If $v$ be the velocity of light in empty space, reckoned in absolute units (cm. per sec.), the electromagnetic unit charge is $v$ electrostatic units.

The unit of work is the Erg, as before.

The unit of Potential Difference is such that the new unit charge, moving through unit potential difference, does unit work (one erg). As the new unit charge is $v$ times the electrostatic unit charge, the new unit potential difference must be $1/v$ electrostatic units.

The value of $v$ can be taken, without sensible error, as $3 \times 10^{10}$ cm. per second. Using E.S. and E.M. as contractions for Electrostatic and Electromagnetic, we therefore have

$$\text{E.M. unit charge} = 3 \times 10^{10} \text{ E.S. unit charge.}$$
$$\text{E.M. unit P.D.} = 1/3 \times 10^{10} \text{ E.M. unit P.D.}$$

**83. Derived Units.**—The unit capacity is that of a condenser whose plates are at unit P.D. when its charge is unity.

Hence the E.M. unit condenser has charge $3 \times 10^{10}$ E.S. units when its P.D. is $1/3 \times 10^{10}$ E.S. units. For unit P.D. (E.S.) its charge would be $9 \times 10^{20}$ (E.S.). It therefore has capacity $9 \times 10^{20}$. Hence

$$\text{E.M. unit capacity} = 9 \times 10^{20} \text{ E.S. units.}$$

The E.S. capacity of a sphere in empty space is equal to its radius. Hence the E.M. unit of capacity is equal to the capacity of a sphere whose radius is $9 \times 10^{20}$ cm. : which is nearly 60,000,000 times the radius of the earth's orbit.

The unit of current, in both systems, is such that unit charge is conveyed per second. Hence

$$\text{E.M. unit current} = 3 \times 10^{10} \text{ E.S. units.}$$

**Ex. 1.**—A machine gun sends 10 pellets per second, of diameter 1 cm., charged to a potential of 20 E.S. units. Find the electric current and express it in electromagnetic units.

The charge carried per pellet

$$= \text{Capacity} \times \text{Potential} = \tfrac{1}{2} \times 20 = 10 \text{ E.S. units.}$$

Hence the current

$$= 10 \times 10 = 100 \text{ E.S. units}$$
$$= 100/3 \times 10^{10} = \tfrac{1}{3} \times 10^{-8} \text{ E.M units.}$$

**Ex. 2.**—Regarding the earth as a conducting sphere isolated in space, express its capacity in electromagnetic units. Its radius can be taken as 6370 kilometres.

$$\text{Capacity} = 6370 \times 1000 \times 100 = 6{\cdot}37 \times 10^{8} \text{ E.S. units (cm.).}$$
$$\text{This} = 6{\cdot}37 \times 10^{8}/9 \times 10^{20} = 7{\cdot}1 \times 10^{-13} \text{ E.M. units.}$$

**84. Ionisation.**—Many chemical compounds, when dissolved in a suitable solvent, partially disintegrate into constituents oppositely charged. Such disintegration is called Ionisation.

Let HCl denote a molecule of hydrochloric acid. Many of these molecules dissociate in aqueous solution into the constituent atoms H and Cl. But the H atom carries a definite positive charge, and the Cl atom an equal minus charge ; we signify this by writing

$$\text{HCl} = \text{H}_+ + \text{Cl}_-.$$

In the same way the dissociation of nitric acid is represented by

$$\text{HNO}_3 = \text{H}_+ + (\text{NO}_3)_- ,$$

the second constituent being not an atom but a group equivalent to an atom.

Such charged atoms or groups are called Ions. They are monovalent if chemically replaceable by a recognised monovalent atom, such as H or Cl. Every monovalent ion carries numerically the same charge ($+^{ve}$ or $-^{ve}$). Divalent ones carry double the charge; we can therefore write

$$H_2SO_4 = 2H_+ + (SO)_{--}$$
$$CuSO_4 = Cu_{++} + (SO_4)_{--}$$

where $--$, $++$, signify double the monovalent atomic charge.

Similarly ions of higher valency carry proportional charges. This ionisation is apparently due to the fact that the solvent (here taken as water, but similar phenomena occur with other solvents) has a large specific inductive capacity and therefore greatly weakens the electric attraction which helped to bind together the $+^{ve}$ and $-^{ve}$ ion into one molecule.

**85. Conduction in Solutions.**—If there be no electric field except that which is due to the ions, these ions travel equally in all directions. But if a P.D. is maintained in the solution, the $+^{ve}$ ions are impelled towards the region of lower potential, and the $-^{ve}$ ions to the higher potential. There is thus a positive current one way and a negative current the other way, and the net effect is a flow of electricity from higher to lower potentials.

In an electrolytic cell, the field in the liquid is usually due to two conductors of the same metal (*e.g.* platinum) kept at a constant P.D. by methods discussed later. The current in the liquid flows from the one at higher potential, which is called the Anode. This name is given to any conductor from which an electric current enters any body or space considered. We can, for instance, speak of the anode or of an electric spark or discharge. The second conductor is the Kathode.

The electronegative ions, which seek the anode, are Anions; the electropositive ones are Kations.

**86. Electrolytic Reactions.**—Electrolysis is chemical separation due to the action of a current. What happens when an ion reaches its electrode is dependent on chemical relationships.

Sometimes the ion is simply given off; sometimes it reacts with the electrode, sometimes reacts with the liquid. The following are particular cases.

**Platinum or Carbon electrodes;** dilute solutions:

I. $HCl = H_+ + Cl_-$:

At anode $Cl + Cl = Cl_2$; at kathode $H + H = H_2$.

Thus chlorine and hydrogen are given off.

II. $H_2SO_4 = 2H_+ + (SO_4)_{++}$:

At anode $SO_4 + H_2O = H_2SO_4 + O$; $O + O = O_2$;

„ kathode $H + H = H_2$.

Oxygen and hydrogen given off.

III. $NaOH = Na_+ + OH_-$:

At anode $OH + OH = H_2O + O$; $O + O = O_2$;

„ kathode $Na + H_2O = NaOH + H$; $H + H = H_2$.

Thus oxygen and water appears at anode, and hydrogen and NaOH at kathode.

**Fused salts** act somewhat like solutions, *e.g.*

IV. $NaOH$ (fused) $= Na_+ + OH_-$:

At anode $OH + OH = O + H_2O$ as before;

„ kathode Na is deposited, because there is no water present with which it can react.

**Strong solutions** may act differently to weak ones.

V. $H_2SO_4 = 2H_+ + (SO_4)_{--}$:

At anode $SO_4 + H_2SO_4 = H_2S_2O_8 + O$; $O + O = O_2$;

„ kathode $H + H = H_2$.

**Copper electrodes:**

VI. $HCl = H_+ + Cl_-$; $Cl + Cl + Cu = CuCl_2$; $H + H = 2H$.

VII. $CuSO_4 = Cu_{++} + (SO_4)_{--}$:

At anode $SO_4 + Cu = CuSO_4$; at kathode Cu is deposited. Thus copper is dissolved from one electrode and deposited on the other, the solution remaining of the same average strength.

VIII. $H_2SO_4 = 2H_+ + SO_4$:

At anode $SO_4 + Cu = CuSO_4$ as above; at kathode $H + H = H_2$. Hydrogen liberated at kathode, copper dissolved at anode. Of course if the current lasts a long time, the solution of $H_2SO_4$ gets weaker and of $CuSO_4$ gets stronger, till a stage is reached at which Cu begins to be deposited on anode as well as hydrogen.

**87. Gram Molecule, or Mol.**—A Gram Molecule (or Mol) of any element or compound is that number of grams of the substance which is numerically equal to its molecular weight. Thus a molecule of hydrogen has formula $H_2$, and therefore a mol of hydrogen gas is two grams. If ions of hydrogen appear in a solution (§ 86, I., II., etc.), their formula is H simply, therefore a mol of hydrogen ion is one gram only. In the same way the molecular weight of sulphuric acid $H_2SO_4$ is

$$2 + 32 + 4 \times 16 = 98,$$

so a mol of sulphuric acid is 98 gm. Equation II. of § 86 shows that a mol of this acid, if completely ionised, gives 2 mols of hydrogen and a mol of sulphion $SO_4$.

Mols must not be confused with equivalents. The equivalent weight, in simple cases, is the quotient of the atomic weight of the ion by the valency. Thus the mol of chlorine ion, Cl, = 35 gm., and Cl is monovalent, therefore the equivalent weight of chlorine is also 35; the

$$\text{mol of sulphion, } SO_4, = 32 + 4 \times 16 = 96 \text{ gm.},$$

but it is divalent, and therefore the equivalent weight is 96 2 = 48. The mol of chlorine gas, $Cl_2$, is 70 gm.

**88. Commercial Measure of Electric Current. Ampere, Coulomb.**—The above explanation of conductivity of solutions involves the property that the charge of electricity conveyed through any one solution is proportional to the quantity of chemical change thereby produced. Consequently a measurable electrolytic effect can be used to measure charge and therefore current.

The commercial **Coulomb** is that charge which passes from anode to kathode of cell containing dilute silver nitrate, $AgNO_3$, when ·001118 gm. of silver is deposited. The **Ampere** is a current of one coulomb per second.

Silver is monovalent, of atomic weight 107·88 (taking oxygen as 16), consequently a mol is 107·88 gm. and carries $107{\cdot}88 \div {\cdot}001118 = 96500$ coulombs. A mol of every other monovalent ion carries the same charge. A mol of a divalent ion carries twice this charge; a mol of a trivalent ion carries thrice as much, etc. Conversely, one coulomb deposits $1/96500 = {\cdot}00001035$ mol of a monovalent ion, half as much of a divalent, one third of a trivalent, and so on.

The student should remember the numbers 96500, ·00001035. The reason for the definition of the coulomb and ampere will be explained later, when absolute electromagnetic (or E.M.) units are defined (§ 161). For the present it must be understood that the ampere is 1/10 of the absolute E.M. unit of current, and the coulomb is 1/10 of the absolute E.M. unit of charge, very approximately. Hence, using § 82, the coulomb is $3 \times 10^9$ E.S. units of charge. A mol of a monovalent ion carries 9650 absolute E.M. units, or nearly $2{\cdot}9 \times 10^{13}$ E.S. units.

**89. Related Commercial Units. Volt, Farad, Joule.**—The Volt is $10^8$ E.M. units of Potential, it is therefore nearly 1/300 E.S. units. The Farad is the capacity of a condenser whose potential rises 1 volt for a charge of 1 coulomb; it is therefore $10^{-9}$ E.M. units. It would be difficult to construct such a condenser: the common standards are microfarads [M.F.] or millionths of a farad: $10^{-15}$ absolute units. A spherical conductor of 1 M.F. capacity has radius $9 \times 10^5$ cm. Taking the earth's radius

as $6{\cdot}37 \times 10^8$ cm., we find its capacity $= 710$ M.F. A **Joule** is the work done when one coulomb alters its potential by 1 volt. It is $10^{-1} \times 10^8 = 10^7$ ergs.

**Ex. 1.**—An accumulator has 20 ampere hours capacity. How much copper will it deposit from copper sulphate?

Note that Capacity is not used in the sense of the definition in § 53: but it simply denotes the quantity of electricity that can be stored in the accumulator. An ampere (one tenth unit current) in a second deposits ·00001035 gram molecule; in 20 hours it deposits $\cdot00001035 \times 20 \times 60 \times 60 = \cdot7452$ gm. molecule. In copper sulphate, $CuSO_4$ Cu is divalent; its atomic weight is 63, hence 63 gm. Cu is equivalent to 2 gm. hydrogen. Hence the Cu deposited is $63 \times \cdot7452 \div 2 = 23{\cdot}5$ gm.

**Ex. 2.**—The same accumulator when run down is recharged by a battery of 4 bichromate cells in series. How much zinc is used up?

In each cell 20 amp. hr. passes; and 65 of Zinc is equivalent to 63 of copper. Hence a similar calculation to the above shows 24·2 gm. is used in each cell; 97 in the four cells.

**Ex. 3.**—A sphere of 1 cm. radius contains a 1 % solution of HCl. If it were possible to withdraw one-hundredth of the Cl atoms with their associated charges, find the E.S. charge and potential of the sphere, and their values in Practical E.M. units. Find also the field (in volts per cm.) at the surface of the sphere.

$$\text{Volume} = \frac{4}{3}\pi, \quad \text{mass of HCl present} = \frac{4\pi}{300} \text{ gm.}$$

Atomic weights are taken as H = 1, Cl = $35\frac{1}{2}$,

$\therefore$ mass of H $= 4\pi/300 \times 36\frac{1}{2} = 8\pi/21900$. The charge required is that which is associated with 1/100 of this hydrogen; it $= 8\pi \times 965/21900$ coulombs $= 1{\cdot}133$ coulombs $= \cdot1133$ E.M. absolute units. This $= 3{\cdot}40 \times 10^9$ E.S. units. The Potential (in E.S. units) = Charge/Radius; and Field = Charge/(radius)$^2$ is $3{\cdot}4 \times 10^9$ E.S. units. Reducing to E.M. units, the potential is

$$3{\cdot}4 \times 10^9 \times 3 \times 10^{10} = 1{\cdot}02 \times 10^{20} \text{ E.M. absolute,}$$
$$= 1{\cdot}02 \times 10^{12} \text{ volts.}$$

The Field, measured by the change of potential per unit distance, happens to be numerically equal to the potential, and therefore is $1{\cdot}02 \times 10^{12}$ volts per cm.

**Ex. 4.**—A certain current deposits 1 gm. of copper from $CuSO_4$ in 2 hours. Find how much hydrogen, by volume, it releases per minute from dilute acid, the temperature and pressure being normal (0° C. and 760 mm. of mercury). Assume 1 c.cm. is ·0000900 gm.

**Ex. 5.**—Show that 1 ampere in 1 second liberates ·115 c.cm. of hydrogen or chlorine at normal temperature and pressure.

**Ex. 6.**—Find the current which liberates 32 c.cm. of hydrogen in 10 minutes. The gas is collected over water, and may be assumed saturated with water vapour. The temperature is 20° C., saturation pressure of water vapour for this temperature is 17 mm., and the pressure in the vessel in which the gas is collected is 75·5 cm.

**Ex. 7.**—Show that enough electricity to charge the earth to a potential of 300 volts is carried by less than 1/4000 c.cm. of hydrogen.

**90. The Conductivity of a Solution.**—Let $m$ = number of mols (gram molecules) of solute in 1 litre of solution and let $k$ = that fraction which is ionised. If we consider the simplest case, a solute like HCl or $HNO_3$, consisting of two monovalent ions, there are $km$ mols of the positive ion and $km$ of the negative; and $(1 - k)m$ of the solute not decomposed, and electrically inert. Consider a tubular portion, $LMM'L'$, of the solution along which the current is flowing:

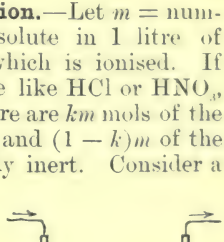

Fig. 42.

Let $A$ be its cross section; let $u$ be velocity of the $+^{ve}$ ion, and $v$ of the $-^{ve}$ ion; then in one second as many gram molecules of the + ion will flow across a section ($PP'$) of area $A$ as are found in a volume $Au$ c.cm.; this number is $Aukm.10^{-3}$. If $q$ = the charge, in absolute E.M. units carried per mol of monovalent ion, the current carried by the + ions across the barrier is $Aukmq.10^{-3}$, and the − ions carry $-Avkmq.10^{-3}$ the other way. Hence the total current

$$= C = A(u + v)kmq.10^{-3}.$$

Let $F$ = field strength = change of potential per unit distance in direction of current. It is found by experiment that ordinarily $C \propto F$. We shall call $C/A$ the specific conductivity, and denote it by $1/R_0$, $R_0$ being called the Specific Resistance.

Hence

$$1/R_0 = 10^{-3}mqk(u + v)/F.$$

**91. Velocities of Ions.**—For a given solution, if only $C$ and $F$ vary, the quantities $m$, $q$ and $k$ are constant. And since $C \propto F$, $R_0$ is also constant. Hence

$$u + v \propto F,$$

and since this is true for all solutions we can infer that $u$ and $v$ separately vary as $F$. But $u$ is the velocity of an ion carrying a constant charge, and $F$ is the force per unit charge, hence the ions travel at a rate proportional directly to the force exerted on them by the electric field.

Now if a body be impelled by a constant force, but have its motion opposed by a resisting medium, the resistance due to the medium always increases as the velocity increases, but vanishes when the velocity is zero. As the velocity approaches the value for which the resistance is equal to the impelling force, the acceleration becomes infinitesimal; and therefore the ultimate motion is with this limiting velocity. The limiting velocity is reached in an exceedingly short time when a very minute body moves in an ordinary resisting medium.

The ions will travel, sensibly, with this limiting velocity. Hence the resistance = the impelling force. But $u \propto F$, therefore we are justified in inferring that the resistance due to the fluid is proportional to the velocity.

**92. Nature of the Resistance to the Ionic Motion.**—The resistance to such swift bodies as meteorites is proportional to (velocity)$^3$. The resistance to certain much slower projectiles, *e.g.* cricket balls, is more or less proportional to (velocity)$^2$. The principal physical cause of the resistance here is that the ball has to give momentum to the air it encounters in order to clear it away. But the resistance to the slowest bodies, minute water drops in air, sand grains in water, or small shot falling in glycerine, is proportional to the velocity itself. This resistance is due to fluid friction, or viscosity.

We infer, the force resisting ionic velocities is also due to viscosity.

Water (and most liquids) gets less viscous as the temperature rises. This implies that the resisting force is

less at higher temperatures for the same velocity. Consequently $(u + v)/F$ is greater, hence the Specific Conductivity of aqueous solutions should increase as the temperature rises. Indeed, if we use solutions so dilute that practically all of the solute is ionised (so that $k = 1$ nearly, and can be treated as a constant) the Specific Conductivity is found to remain inversely proportional to the viscosity at all ordinary temperatures.

**93. Temperature Coefficient of Resistance.** *Definition.*—The temperature coefficient of any physical quantity is the amount by which it increases per 1° C. rise of temperature, divided by its value at some standard temperature which usually is taken 0° C., the freezing point of water. Thus the temperature coefficient of Resistance is the increase of resistance per 1° C. rise of temperature, divided by the resistance at 0° C.

The temperature coefficient of the resistance of a dilute solution is equal to the temperature coefficient of the viscosity of solvent (usually water).

Since, however, the resistance is proportional not only to $F/(u + v)$ but also to $k$, the equality between the temperature coefficients of resistance and of viscosity only holds when $k$ is constant. It holds approximately in very dilute solutions, for $k = 1$. It breaks down in dense ones.

**94. Molecular Conductivity of Salts.**—The specific conductivity $= 10^{-3}\ mqk\ (u + v)/F$. Now $u$ and $v \propto F$, hence we can put $u/F = u_0$, $u/F = v_0$, these being the velocities under unit potential gradient. Hence the specific conductivity

$$= 10^{-3}\ mqk\ (u_0 + v_0).$$

If $k$ were constant, this would $\propto m$. Its quotient by $m$,

$$10^{-3}\ kq\ (u_0 + v_0),$$

is called the Specific Molecular Conductivity. It is the conductivity divided by the strength of the solution in gram molecules per litre. It is nearly constant for dilute solutions, but alters considerably with $k$ when the solutions

are dense. The only varying letter is $k$, and therefore the changes of molecular conductivity exhibit the amount of dissociation. In fact the dissociation factor $k$ of a salt

$$= \text{Molecular conductivity} \div \text{The limit approached by the molecular conductivity when the concentration is indefinitely small.}$$

The dissociation factor can be investigated by other methods, for instance by the change of boiling point produced by the dissolved substance. These methods agree with the electrical methods, both with respect to the behaviour of salts and the peculiar property of acids, noted in next paragraph.

**Ex. 1.**—A very dilute solution of hydrochloric acid has composition ·001 gm. molecule per litre. Its specific conductivity at 18° is $3453 \times 10^{-16}$ absolute C.G.S. units. Find the sum of velocities of the hydrogen and chlorine ions for a potential gradient of 1 volt per cm.

$$\text{Here } 3453 \times 10^{-16} = 10^{-3}\, mqk\,(u + v)/F.$$

But $m = ·001 = 10^{-}$ , $q = 9650$, $k$ can be taken $= 1$, and since 1 volt $= 10^{8}$ absolute units, $\therefore F = 10^{8}$. Hence $u_0 + v_0 = ·00358$.

**Ex. 2.**—Consider similarly a solution of potassium chloride, ·00001 gm. molecule per litre, conductivity $1293 \times 10^{-18}$, potential gradient 1 volt per cm.

$$u_0 + v_0 = ·00134.$$

**Ex. 3.**—Consider solution of ·224 gm. potassium chloride in 1 litre; conductivety $353 \times 10^{-15}$.

The atomic weight of KCl is $39 + 35·5 = 74·5$, therefore the number of gram molecules per litre is $·224 \div 74·5 = ·03$. We get $u + v_0 = ·0012$.

**Ex. 4.**—A 5 per cent. solution of $CaCl_2$ (*i.e.* 5 gm. of calcium chloride to 100 gm. solution) has specific conductivity ·0643. Show that the molecular conductivity is ·143. Note that calcium is divalent, and has atomic weight 40. Assuming $k = 1$, find the value of $u_0 + v_0$.

**Ex. 5.**—The velocities (per unit potential gradient) of Na and Cl in water at 18° are ·00045, ·00068. Calculate the molecular conductivity at infinite dilution, and estimate the specific conductivity of a solution of 1 gm. per litre.

**95. Solutions which behave anomalously.**—The behaviour of dilute salts, such as NaCl, $KNO_3$, etc., agrees with the above theory so well that it has been found possible to draw up a table of ionic velocities (per unit

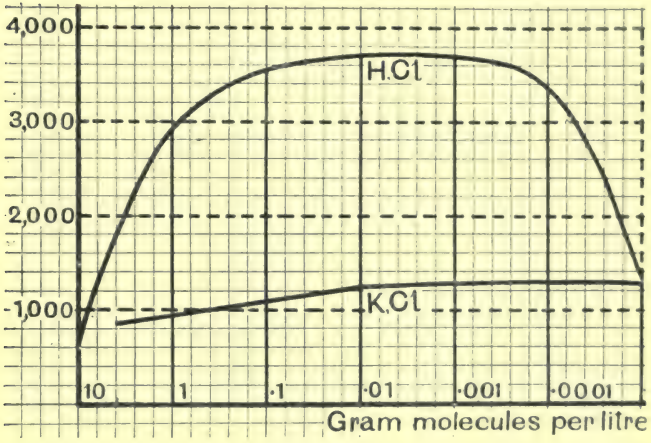


Fig. 43.

Abscissae are proportional to the logarithms of the dilutions: ordinates represent the Specific Molecular Conductivity $\times 10^{14}$. The Specific Conductivity of the Solution is deduced by multiplying the Specific Molecular Conductivity by the number of gram molecules per litre. A further multiplication by $10^{-9}$ reduces to Practical Units.

potential gradient) from which conductivities of dilute solutions can be calculated with fair accuracy.

But acids, like HCl, $HNO_3$, show curious divergences which are not yet explained. As the solutions are made more dilute, the molecular conductivity first approaches the value demanded by theory, but afterwards diverges from it widely. See Fig. 43.

Some solutions (*e.g.* sugar or alcohol in water) do not appear to ionise, and do not show electrical conductivity.

**96. Ratio of Ionic Velocities.**—If we consider an imaginary barrier across the liquid, the number of gram molecules of $+^{ve}$ ion that cross with the current per second is $10^{-3}\ Amu$, and of $-^{ve}$ ions that cross against the current is $10^{-3}\ Amv$; the ratio of these is $u : v$. Hence if all quantities be measured in gram molecules the depletions of the solution on the two sides of the barrier are as $u : v$; if this be not modified by other chemical charge. Take for instance a solution of copper sulphate $CuSo_4$ in a cell with platinum electrodes. For every $u$ mols of Cu that cross the barrier towards the anode, $v$ mols of $SO_4$ cross towards anode; each of the ions being divalent. The solution on the anode side is weakened by $u$ mols of $CuSO_4$, destroyed to furnish the Cu that has migrated; and the solution on the kathode side is weakened by $v$ mols of $CuSO_4$. Of course the $v$ liberated mols of $SO_4$ on the anode side form sulphuric acid, liberating oxygen (§ 86, II.), and the $u$ mols of Cu on the kathode side are deposited.

**97. Ionic Velocities in Copper Electrolytic Cell.**—Here we have copper electrodes. Consider the anode side of the barrier. For every $u$ mols of Cu which cross the barrier, $u$ mols of $SO_4$ are left behind, and $v$ mols of $SO_4$ enter the solution through the barrier. Hence there is an excess of $u + v$ mols of $SO_4$, which attacks the anode. The anode loses $u + v$ of Cu, and the solution on the anode side is stronger by $v$ mols of $CuSO_4$ in virtue of the $v$ of $SO_4$ which has entered and attacked the anode. Consider next the kathode side. The $u$ mols of Cu which enter are deposited on the kathode. The $v$ mols of $SO_4$ which go away weaken the solution by $v$ mols of $CuSO_4$, leaving $v$ mols of Cu to be deposited. Hence $u + v$ mols of Cu is taken from the anode and deposited on kathode; while the solution is strengthened near the anode, and weakened near the kathode, by $u$ mols.

**98. First Method of Observing Ratios of Ionic Velocities.**—To observe $u : v$. A vessel is constructed as in Fig. 44. A silver anode $A$ is at bottom of a burette; $B$ is a silver kathode. The solution is silver nitrate $AgNO_3$; the ions Ag and $NO_3$ are monovalent. The composition of solution is first found and plate $B$ weighed. Then a current is allowed to flow for some time, and the solution becomes denser around $A$. By running out half the contents of burette and analysing one can find the strengthening of solutions which $\propto u$, and by finding increase of weight of $B$ we get a number $\propto u + v$; hence $u/v$ is found.

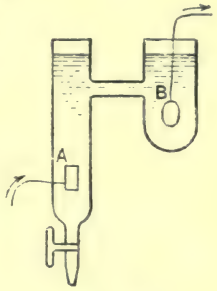

Fig. 44.

**99. Second Method of Observing Ratios of Ionic Velocities.**—The anode and kathode are in vessels connected by a tube which contains a jelly solution. Such salts are chosen that the progress of the ions can be watched by colour changes produced. For instance,* Oliver Lodge used sodium chloride in agar jelly in the tube rendered slightly alkaline to bring out the red colour of a little phenolphthalein added as an indicator. The anode vessel contained an acid, so that the $+^{ve}$ ion was hydrogen. As this travelled along the tube it decolorised the phenolphthalein. The velocity of H for a potential gradient of 1 volt was found to be about ·0026 cm. per sec. Masson used a colourless salt, KCl, in the jelly and solutions of potassium chromate around kathode, and copper sulphate around anode. The copper travelling with the current entered the jelly and

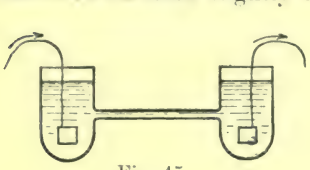
Fig. 45.

* These particulars are from Whetham, *Theory of Solution*.

formed blue copper chloride, the advance of the colour showing the rate at which the potassium ions left the chrome ion available to combine with copper. Since Cu travels less rapidly than K under the same potential difference, there was no tendency to blurring the colour boundary by the Cu getting ahead. In the same way the chrome ion advancing against the current followed the swifter Cl ion and formed yellow potassium chromate in the jelly.

One might imagine that the swifter K and Cl would get away from the pursuing Cu and $CrO_4$. In practice this is not so. The current is physically bound to be continuous. The K would travel faster than Cu under the same potential gradient, and it is physically necessary that it should travel equally fast, consequently the potential must alter more rapidly with the distance in the region of the Cu ion. A steady state is reached in which the potential gradient is least steep in the KCl. Hence the experiment compares the velocity of K and Cl under the same potential gradient, and is not complicated by the velocities of Cu and $CrO_4$ which are travelling under different potential gradients.

Whetham used a similar method, with a vertical liquid column instead of a jelly solution. A porous diaphragm prevented convective motion of the column due to its change of density.

# CHAPTER VI.

## ELECTROMOTIVE FORCE.

**100. Continuous Currents.**—When a Wimshurst is used to produce a shower of sparks between its terminals, we can consider an electric current to flow across the spark gap and around a circuit completed by the machine. A voltaic cell is quite analogous. Its current is usually much larger than that of a Wimshurst when it is working; and the free charges, which are conspicuous in the Wimshurst, are exceedingly small in the cell. But the differences are only of degree, and consist in the fact that there is hardly any local accumulation of electricity. Practically all makes a complete circuit, through the cell and connecting wires, the positive going one way and the negative the other.

We have seen (§ 80) that in the production of a series of sparks from a Wimshurst machine there is a conversion of electrical energy into heat, and that this electrical energy is produced by another form of energy (mechanical) supplied by the person who turns the handle. He also supplies other energy, which is dissipated, *i.e.* is uselessly converted into heat, in overcoming the friction of the moving parts. Similarly a voltaic cell will only deliver a continuous current if sufficient energy be supplied. Some of this energy is dissipated, some is directly used in producing the current, and some can be used in other ways, *e.g.* to produce chemical reactions.

**101. Reversible and Irreversible Operations.**—Chemical operations depending on the flow of a current are usually reversible. In the copper electrolytic cell, *e.g.* (§ 86, VII.), the current removes copper from one plate

and deposits it on the other. A current sent the other way would produce the contrary effect. We shall find later that certain other effects of a current are also reversible. That is to say, what is done by a current is undone by an equal but contrary current.

The flow of a current in any conductor produces a heating effect which is irreversible. That is to say, the current heats the conductor, and an equal but contrary current would also heat the conductor equally.* Chemical operations can be irreversible in their nature.

Take as an example the simple Volta cell. Hydrogen is given off from the copper plate and most of it escapes. It is therefore impossible that this hydrogen should be recombined on reversing the current.

**102. Electromotive Force [E.M.F.].**—The electromotive force of a battery, or other source of electric current, is the work which it provides (in reversible processes) per unit current per second. If a unit current flows for a second, it is evident that a unit charge flows round the current circuit. Hence the E.M.F. can also be defined as the work provided (reversibly) per unit charge.

*Back Electromotive Force.*—This is the work done, reversibly, per unit charge, at the expense of the energy of the current.

*Resultant Electromotive Force* is the work done, irreversibly, per unit charge, at the expense of the energy of the current. Almost the whole of this work is dissipated in the irreversible production of heat.

Since all energy given to the current must be eventually given back either reversibly or irreversibly, it follows that the E.M.F. in any circuit is equal to the sum of the back E.M.F. and resultant E.M.F.

**103. Units of Energy, Power, and E.M.F.**—The absolute unit of energy is the Erg. The practical or engineering unit is the Joule, which $= 10^7$ ergs.

* There also may exist heating effects which are reversible. These are considered later (Chap. XIII.).

**Power** is rate of working. The absolute unit is one erg per second. The practical unit, which is called a Watt, is one Joule per second, and is $10^7$ absolute units. Engineers frequently use a larger unit, called the Kilowatt, which is 1,000 Watts. Founded on the Kilowatt is the Board of Trade [B.O.T.] Unit of Energy. It is the Kilowatt Hour, *i.e.* the work done in 3,600 seconds at the rate of 1 kilowatt; and evidently its value is 3,600,000 joules $= 3{\cdot}6 \times 10^{13}$ ergs.

**Electromotive Force** is work per unit charge. The electro-magnetic Absolute unit is the E.M.F of a source which expends one erg in producing the flow of electro-magnetic unit charge; *i.e.* unit current per second. The Practical unit is called a Volt. It is the E.M.F. of a source which expends 1 joule in producing one coulomb, *i.e.* 1 ampere for 1 second. Thus a volt expends $10^7$ ergs in producing 1/10 unit current for 1 second; it therefore is $10^8$ absolute units.

**104. The Thomson Rule.**—This merely expresses that the E.M.F. of a cell can be calculated from the energy of its chemical reactions. It appears a mere consequence of the definitions of § 102, but historically E.M.F.'s were measured before first Helmholtz and afterwards Sir William Thomson (Lord Kelvin) explained their connection with energy.

The chemical processes in a battery are either electrolytic or secondary reactions due to the electrolysis; consequently all are proportional to the current and to the time. That is, the quantity of every reaction is proportional to the charge that has gone round the circuit. If the energy of every reaction is calculable, and these reactions are reversible, we know the work supplied per unit; *i.e.* the E.M.F. of the cell.

The energy of chemical processes depend only on the final result. Consequently the total energy can be found from the apparent reactions in a cell, even when these differ considerably from the real ones. And the energy of a reaction can be measured, with more or less certainty, by the heat it produces.

The heats of formation have been calculated for a large number of reactions. It would seem therefore quite simple to substitute and find E.M.F.'s accurately. The difficulties of the method are, first, that we do not always know which reactions provide energy for the current, and which simply produce heat or secondary processes; and, secondly, that we do not fully understand all the reactions that take place. In a Volta cell, besides the production of zinc sulphate and destruction of sulphuric acid, there are differences produced in concentration and in ionisation of solutions; and the work done or provided in these processes is not accurately known.

**105. Heats of Reactions.**—Let $H$ be the heat of a reaction, in calories per gram equivalent; *i.e.* for quantities of the chemicals employed equivalent to 1 gm. of hydrogen. Of course a gram equivalent of a monovalent ion, or of a compound from which a monovalent ion is liberated, is one mol. That of a divalent ion is half a mol.

We give a list of heats of reaction of a few common substances, taken from the *Annuaire du Bureau des Longitudes*, 1910.

To use the table proceed as follows :—One coulomb (or 1 ampere in one second) would release ·00001035 gm. of hydrogen, or involve ·00001035 gm. equivalent of any other substance. Consequently, the reaction of 1 gm. equivalent occurs when 96500 coulombs flow round the circuit. The energy provided is $H$ calories $= 4{\cdot}18\ H$ joules. Consequently the E.M.F., in so far as it depends on the reaction, is $4{\cdot}18\ H \div 96500 = H/23{,}000$ nearly. The questions remain how far the E.M.F. does actually depend on the reaction considered, and how much other reactions contribute; and if these questions are not answered correctly the calculated E.M.F. will be wrong.

The tabulated reactions are per gm. molecule, not per gm. equivalent. This is because the same substance can enter into reactions involving different valences. Compare $H_2O + K = KHO + H$, and $H_2O + Zn = ZnO + 2H$. In the first case 18 gm. of water (1 mol) give 1 gm. of hydrogen, so the gram equivalent is the gram molecule.

In the second, 18 gm. of water give 2 gm. hydrogen, so the gm. equivalent is half the gm. molecule. In order to get $H$ divide by the valency proper to the reaction considered. Divide again by 23 for the energy in joules per coulomb, *i.e.* the E.M.F.

Heats of formation in thousands of calories.

| | | |
|---|---|---|
| $H_2O$ | 69 | (liquid) |
| $HCl$ | 39·4 | (dilute) |
| $ZnCl_2$ | 113 | ,, |
| $CuCl_2$ | 62·5 | ,, |
| $AgCl$ | 29 | ,, |
| $H_2SO_4$ | 210·2 | ,, |
| $ZnSO_4$ | 229·6 | ,, |
| $CuSO_4$ | 181·7 | ,, |
| $ZnO_2$ | 84·8 | ,, |

**Ex. 1.**—Find the E.M.F. of a Daniell Cell, composed of Zn, $ZnSO_4$, $CuSO_4$, Cu.

The final process is that $ZnSO_4$ is formed, $CuSO_4$ destroyed. The heats of formation of these salts are 229600 and 181700. Both are divalent, consequently the numbers per gm. equivalent are 114800 and 90850. Divide the difference by 23000 (or the difference of the original numbers by 46000); the E.M.F. = 1·04.

This is not quite accurate, in fact 1·08 is the ordinary E.M.F. There are other processes besides the chemical ones; thus the $ZnSO_4$ solution gets less dense and the $CuSO_4$ solution denser. The heats of solution ought to be considered, and a possibility always exists of heat being reversibly disengaged or absorbed at an electrode.

**Ex. 2.**—What is the H.P. of a Daniell consuming 15 gm. of zinc per hour, if the E.M.F. be 1·08?

The quantity of electricity that passes per 1 gm. equivalent per second is 96500 coulombs, and a gm. equivalent is $32\frac{1}{2}$ gm. In 1 sec. the zinc consumed is $15/3600 = 1/240$ gm.; $\therefore$ the current $= 96500/240 \times 32\frac{1}{2}$ amp. $= 9650/780$. The energy supplied per sec. $= 9650 \times 1{\cdot}08/756$ joules, and the H.P. is $9650 \times 1{\cdot}08/780 \times 746 = {\cdot}019$.

**Ex. 3.**—Find E.M.F. needed to decompose water, with platinum electrodes.

The final reaction is $2H_2O = 2H_2 + O_2$. Heat of formation per mol of $H_2O$ is 69 in thousands. The E.M.F. is $69/2 \times 23 = 1{\cdot}5$ volts.

**Ex. 4.**—Find E.M.F. needed to decompose HCl into Hydrogen and Chlorine.

$$\text{E.M.F.} = 39{\cdot}4 \cdot 23 = 1{\cdot}7.$$

**Ex. 5.**—Find E.M.F. of a silver chloride cell.
The reaction is $2AgCl + Zn = ZnCl_2 + 2Ag$. Heat of silver chloride is 29 ; that of $ZnCl_2$, divide by the valency 2, is 56·5. Difference is 27·5 $\therefore$ E.M.F. = 27·5/23 = 1·2.

**Ex. 6.**—Find the E.M.F. needed to decompose zinc chloride.

**Ex. 7.**—Find the E.M.F. of a cell composed of Zn, $ZnCl_2$, $CuCl_2$, Cu.

**Table of E.M.F.'s of common Cells at 16°.**

| Name. | Constituents. | | | | E.M.F. |
|---|---|---|---|---|---|
| Standard cells. | | | | | |
| Daniell | Zn | $ZnSO_4$ | $CuSO_4$ | Cu | 1·0 |
| Clark | Zn | $ZnSO_4$ | $Hg_2SO_4$ | Hg | 1·434 |
| Cadmium | Cd | $CdSO_4$ | $Hg_2SO_4$ | Hg | 1·019 |
| Common cells. | | | | | |
| Leclanché | Zn | $NH_4Cl$ | $MnO_2$ | C | |
| Bunsen | Zn | $H_2SO_4$ | $HNO_3$ | C | |
| Grove | Zn | $H_2SO_4$ | $HNO_3$ | Pt | |

**106. Application of Second Law of Thermodynamics.**—Consider a cell entirely reversible. When one coulomb passes, let $W$ be the energy supplied by the chemical processes, and $H$ be any energy that has to be supplied in the form of heat in order to keep the cell at a constant temperature $\theta$ absolute. $H$ and $W$ are measured in joules ; hence if $E$ be the E.M.F. in volts,

$$E = W + H.$$

At a lower temperature $\theta'$ let the E.M.F. be $E'$. The chemical processes are unaltered, so $W$ is the same, and if $H'$ be the heat supplied,

$$E' = W + H'.$$

Hence $$E - E' = H - H'.$$

If the cell at $\theta'$ be connected, reversed, to an identical cell at $\theta$ and one coulomb passes, then $H$ joules of heat are taken in at the higher temperature and $H'$ given out at the lower temperature, hence the thermal efficiency is $(H - H')/H$. By thermodynamic principles, this

$$= (\theta - \theta')/\theta.$$

Hence $$H/\theta = (H - H')/(\theta - \theta') = (E - E')/(\theta - \theta')$$
$$= dE/d\theta,$$

if the charges be taken indefinitely small.

By this equation we can calculate $H$, the heat reversibly absorbed per coulomb.

**Example** (from Arrhenius).—Small electrolysed dilute acid between electrodes of palladium, which absorb most of the separated gases. The operation is therefore reversible. The E.M.F. given by Thomson's Rule is (§ 105, Ex. 3) 1·5 volts. But the observed E.M.F. was only 1·08; and the change of E.M.F. per 1° rise was − 1/700 at 20°.

Using $H/\theta = -1/700$, and $\theta = 273 + 20 = 293$, we get $H = -293/700 = -·42$.

Hence although 1·5 joules are provided by the chemical reactions per coulomb, ·42 appear as heat, and the E.M.F. should be 1·08 as observed.

**107. Irreversible Processes.**—The most universal and typical irreversible process is the production of heat. If a Daniell cell and a coil of wire forming a circuit be enclosed in an ice calorimeter which keeps it at constant temperature, the melting of the ice will show a liberation of heat. After such a time that one coulomb has passed, there will be definite chemical changes and definite heat liberated, and no other process in which work is done. Consequently the heat is exactly as great as if it had been produced directly by the chemical processes without an electric current. If some outside agency be used to send an equal current an equal time through the cell in a reverse direction, the chemical changes will be undone again, but practically as much heat is produced again.

The simplest experiment is on a metal wire, for no chemical change occurs.

If a current be sent through a metal wire at uniform temperature, the heat produced per second is proportional to the square of the current. Hence if $R$ be the heat in energy units produced per unit current per second, the heat produced by current $c$ in time $t$

$$= Rc^2t.$$

**108. Resistance.**—The **resistance of a wire** (whose temperature is uniform) is the heat energy produced per second when unit current passes through it.

The *absolute unit of resistance* is the resistance of a wire in which unit current [§§ 82, 161] produces 1 erg of heat in 1 second.

The *Ohm* or *Practical unit of resistance* is the resistance of a wire in which 1 ampere produces 1 joule of heat in 1 second.

Hence an ohm is the resistance of a wire in which a current 1/10 absolute units produces $10^7$ ergs. If it be $R$ absolute units,

$$\therefore\ 10^7 = R\,(1/10)^2\ ;\ R = 10^9.$$

That is,

$$1 \text{ ohm} = 10^9 \text{ E.M. absolute units.}$$

The **resistance of any conductor** is the heat energy produced per second by **irreversible processes** when unit current passes through it.

**109. Resistances in Series.**—If the resistances be in series, the same current $c$ goes through all. If $R$ be joint resistance, and $R_1$, $R_2$, $R_3$ the separate resistances, then the heat $H$ produced in time $t$ (§§ 107, 108)

$$= Rc^2$$

and

$$= R_1c^2 + R_2c^2 + R_3c^2 + \ldots$$

$$\therefore\ R = R_1 + R_2 + R_3 \ldots$$

Hence resistance in series are additive.

**110. Ohm's Law.**—If $E$ be the total E.M.F. of a circuit (the algebraic sum of E.M.F.'s and back E.M.F.'s) and $t$ be current, then $Ect$, is the total energy provided in time $t$ by all the reversible processes.

The energy provided in the irreversible production of heat

$$= Rc^2t$$

$$\therefore\ Ect = Rc^2t$$

$$\therefore\ E = Rc$$

or, in words,

The E.M.F. (resultant) is the product of the current and the resistance.

The equation holds whether $E$, $R$, $c$ are measured in absolute units or in the practical system, volts, ohms, and amperes.

**Ex. 1.**—A wire carrying a current of ·5 ampere is immersed in an ice calorimeter and melts 1 gm. per minute. Find its resistance in ohms.

To melt 1 gm. of ice requires 80 calories = 80 × 4·18 = 334·4 joules. The time $t$ = 60 sec. and $C$ = ·5 amp.

$$\therefore 334{\cdot}4 = R(\cdot 5)^2 60$$
$$R = 22{\cdot}3 \text{ ohms.}$$

**Ex. 2.**—An incandescent light takes a current of ·1 amp. and uses energy at rate 1·6 watts. Find its resistance.

In one second it uses 1·6 joules, this therefore is heat produced.

$$1{\cdot}6 = R \times (\cdot 1)^2, \qquad R = 160 \text{ ohms.}$$

**Ex. 3.**—If we reckon metal filament lamps to use 1·2 watts per candle power, find the resistance of a 60 candle power lamp which uses 1/3 ampere.

In a second it produces 72 joules.

$$\therefore 72 = R(1/3)^2 \; ; \; R = 648 \text{ ohms.}$$

**Ex. 4.**—A calorimeter contains 80 gm. of water, and its water equivalent is 3 gm. What rise of temperature should we expect if 1·2 amperes flow for 2 minutes through an immersed coil of wire whose resistance is 1 ohm?

$$H = Rc^2t = 1\ (1{\cdot}2)^2 120 = 172{\cdot}8 \text{ joules.}$$

Thermal capacity = 83, *i.e.* to raise its temperature 1° it needs 83 calories, or 83 × 4·18 joules.

$$\therefore \text{Rise of temperature} = 172{\cdot}8/83 \times 4{\cdot}18 = \cdot 5^\circ.$$

**Ex. 5.**—A current $c$ is divided between two wires of resistance $R_1$ and $R_2$. Find what the current in each wire must be if the heat produced per second be a minimum.

Let $x$ flow along $R_1$, and $c - x$ along $R_2$.

Then $H = R_1 x^2 + R_2(c - x)^2$

$$= (R_1 + R_2)x^2 - 2R_2cx + R_2c^2$$
$$= (R_1 + R_2)\{x - R_2c/(R_1 + R_2)\}^2$$
$$+ R_1R_2c^2/(R_1 + R_2).$$

The first term is an exact square and cannot be negative, hence it is a minimum when it is zero.

Hence

$$x = R_2c/(R_1 + R_2),$$
$$c - x = R_1c/(R_1 + R_2)$$

and

$$H = R_1R_2c^2/(R_1 + R_2);$$

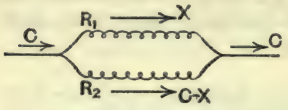


Fig. 46.

therefore the current divides in inverse ratio of the resistances, and the joint resistance is $R_1R_2/(R_1 + R_2)$. See § 112.

**Ex. 6.**—In a single Daniel cell ($E = 1{\cdot}08$) find the ratio of the calories of heat produced to the grams of zinc consumed.

If $H$ be in calories, $4{\cdot}18\ H$ is in joules,

$$\therefore 4{\cdot}18\ H = Rc^2t.$$
$$\text{But } E = Rc = 1{\cdot}08$$
$$\therefore 4{\cdot}18\ H = 1{\cdot}08ct.$$

But 1 gram equivalent of zinc, or 32·5 gm., is consumed for every 96500 coulombs. If $\therefore$ $x$ gm. be consumed,

$$ct = 96500\ x\ /32{\cdot}5$$
$$\therefore H/x = 1{\cdot}08 \times 96500/4{\cdot}18 \times 32{\cdot}5$$
$$= 770.$$

**Ex. 7.**—If $n$ Daniell cells in series be used, find ratio of heat produced to zinc consumed.

$$\text{Here } E = 1{\cdot}08n, \ \therefore 4{\cdot}18\ H = 1{\cdot}08nct.$$

But every 96500 coulombs traverses $n$ cells in series and consumes 32·5 gm. of zinc in each,

$\therefore nct = 96500 \times 1{\cdot}08/32{\cdot}7$ and $H/x = 770$, the same as before.

**Ex. 8.**—Compare the expense of electric heating by use of Daniell cells with that of heating by burning zinc.

1 gm. zinc in a cell gives 770 calories.

1 gm. zinc burned to $ZnO_2$ gives $84800 \div 65{\cdot}4 = 1300$.

Hence it is 1·7 times cheaper to burn zinc as fuel.

**Ex. 9.**—Find in calories the heat developed per minute by a current of ·5 ampere in a resistance of 8 ohms.

**Ex. 10.**—An electric flatiron has to develop heat at the rate of 20 calories per second. The E.M.F. is 120 volts; find the resistance (supposed all to be within the iron).

**Ex. 11.**—A lamp, supplied with current at 200 volts, gives heat energy at the rate of 60 watts. Find the current.

**Ex. 12.**—An unknown current, passed through a standard ohm resistance, supplies energy 50 calories per minute. Find the current.

# CHAPTER VII.

## PROBLEMS ON FLOW OF ELECTRICITY IN CONDUCTORS.

**111. Extension of Ohm's Law, I.**—Consider a homogeneous conductor at uniform temperature. Imagine it contained in a calorimeter, current being led to it from without. The heat produced is $Rc^2t$. No change is supposed to occur in the conductor, consequently there is no E.M.F. nor back E.M.F., hence the above energy can only be drawn from that of the current.

But if the terminals by which the current enters and leaves have a potential difference $V$, we know (§ 11) that every unit charge does $V$ units of work in passing from one to the other. A total charge, $ct$, has passed, and therefore the energy available is $Vct$. Assuming this is converted into the above heat,

$$Vct = Rc^2t,$$
$$\therefore V = Rc.$$

That is: In a homogeneous conductor (all at same temperature) Potential Difference = Current × Resistance.

Ohm regarded potential difference as External E.M.F., taking what we have defined as E.M.F. to be Internal E.M.F. The common experimental proof of Ohm's law is a verification that $V \propto c$ for the same conductor; $V$ being measured by an electrometer and $c$ by a galvanometer.

**Ex. 1.**—A battery has resistance ·5 ohm. Its terminals are joined to a quadrant electrometer and produce deflection 30 on scale. What deflection is produced when they are connected also by a wire of resistance 5, 4, 3, 2, 1 ohms?

If $B$ be battery resistance, $R$ resistance of connecting wire, then $B + R$ is total resistance ; $\therefore c = E/(B + R)$.

If $V$ be P.D. of ends of wire, $V = Rc$, $\therefore V = ER/(B + R)$.

When $R$ is $\infty$, $R/(B + R) = 1 - B/(B + R) = 1$, $\therefore V = E$.

So when no wire is inserted the electrometer reads the E.M.F. of battery.

In our example $B = \frac{1}{2}$ ; if $\therefore$ R = 5, 4, 3, 2, 1 we have $R/(R + B) = \frac{10}{11}, \frac{8}{9}, \frac{6}{7}, \frac{4}{5}, \frac{2}{3}$. Since the whole E.M.F. is indicated by 30 divisions, the deflection in the first case is $30 \times 10/11 = 27{\cdot}3$, and in the other cases 26·7, 25·7, 24, 20.

**Ex. 2.**—A Voltmeter (see § 176) can be taken as simply an instrument to measure P.D. Find the resistance of a battery of accumulators, given that the voltmeter reads 8 volts at its terminals, and this drops to 7 volts when they are connected by a 14 ohm wire.

$$E = 8,\ ER/(B + R) = 7,\ R = 14.$$

Hence
$$\frac{14}{B + 14} = \frac{7}{8};\ B = 2 \text{ ohms}.$$

**Ex. 3.**—An Ammeter (§ 176) is supposed simply to read current. If an ammeter in series with a lamp show $\frac{1}{4}$ ampere, and a voltmeter connected to its terminals shows 180 volts, find its resistance.

$$180 \div \tfrac{1}{4} = 720 \text{ ohms}.$$

**Ex. 4.**—Find the current in a uniform wire of 5 ohms resistance, a voltmeter across its terminals showing 4 volts. Find also the potential difference per cm. length of wire, the wire being 200 cm. long.

Current is 4/5 amp. Resistance of each cm. is 1/40 ohm.
Hence P.D. = 1/50 volt.

**Ex. 5.**—A current ·02 amp. passes along a uniform trough of copper sulphate. Two wire gratings connected to terminals of an electrometer are dropped into the trough 20 cm. apart ; and show a P.D. of ·6 volt. Find the resistance of the trough per cm. length.

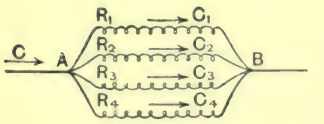

Fig. 47.

Resistance of 20 cm. = ·6 ÷ ·02 = 30 volts ; resistance per cm. is 1½ volts.

**112. Conductors in Parallel.**—To find the joint resistances of two or more conductors whose terminals are connected where the current enters, and where it leaves, the system (Fig. 47).

Let $R_1$, $R_2$ . . . . be their resistances, and $R$ the joint resistance. Let $V$ be P.D. between their terminals $A$ and $B$; $c$ the total current; $c_1$, $c_2$ . . . currents in $R_1$, $R_2$, . . . .

Then

$$V = cR = c_1R_1 = c_2R_2 = \text{etc.}$$

$$\therefore\ c = V/R,\ c_1 = V/R_1,\ c_2 = V/R_2, \text{ etc.}$$

But

$$c = c_1 + c_2 + \text{etc.}$$

$$\therefore \text{ dividing out } V$$

$$1/R = 1/R_1 + 1/R_2 + \text{etc.}$$

$1/R$ is called the Conductivity. It is measured in Mhos if $R$ be in Ohms. Evidently the total conductivity of conductors in parallel

= sum of their separate conductivities.

See § 110, Ex. 5, for proof that the above partition of currents is that which produces the least heat in the system.

**113. Extension of Ohm's Law, II. Kirchoff's Law.**—Consider any conductor which forms a portion of a circuit and which includes E.M.F.'s or back E.M.F.'s, either of the types already considered [Batteries and Voltameters] or of other types to be considered in later chapters. Let $E$ be the algebraic sum of the E.M.F.'s in the conductor, and let $V$ be the potential difference between the ends of the conductor.

Then $Rc^2t$ = heat produced by the current in time $t$. Also $Vct$ = work provided (as in previous paragraph) by the charge as it alters its potential, and $Ect$ = work provided by chemical (or other) E.M.F.'s.

Hence

$$Rc^2t = Vct + Ect,$$

$$\therefore\ \mathbf{Rc = V + E}.$$

**114. Currents in a Network of Conductors.**—To avoid considering $V$ for all the points of a complicated system of conductors, one can reduce the system into

separate meshes. Let $ABCDE$ (Fig. 48) be such a mesh, each arm of which may contain E.M.F.'s. Let $E_{AB}$ = algebraic sum of E.M.F. in arm $AB$ reckoning those positive which supply energy to a current going from $A$ to $B$. Let $R_{AB}$ be resistance of $AB$, $c_{AB}$ current in $AB$, $V_A$ potential at $A$.

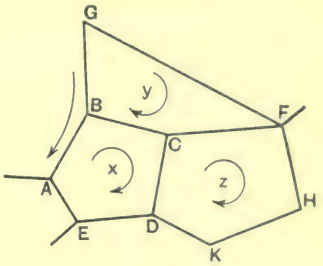

Fig. 48.

Then

$$R_{AB}c_{AB} = V_A - V_B + E_{AB}$$
$$R_{BC}c_{BC} = V_B - V_C + E_{BC}$$
$$R_{CD}c_{CD} = V_C - V_D + E_{CD}$$
$$R_{DE}c_{DE} = V_D - V_E + E_{DE}$$
$$R_{EA}c_{EA} = V_E - V_A + E_{EA}$$

Adding for this mesh, all the $V$'s cancel.

Therefore

$$R_{AB}c_{AB} + R_{BC}c_{BC} + \ldots . R_{EA}c_{EA} = E_{AB} + E_{BC} + \ldots . E_{EA}.$$

Kirchoff enunciated therefore two laws, of which this is the second. They are

(1) Algebraic sum of currents flowing to (or from) any angle of the network is zero.

(2) Algebraic sum of products of current by resistance, taken round any mesh, = algebraic sum of E.M.F.'s round that mesh.

**115. Maxwell's method of simplifying the Currents in a Network.**—Imagine a constant mesh current put round each independent mesh. Thus, in Fig. 48, $x$ may be taken as the mesh current for $ABCDE$, $y$ for $BGFC$, $z$ for $CFHKD$, etc. Of course $ABGFCDEA$ is also a mesh, but is not an independent one, for it is the sum of the meshes $ABCDE$ and $BGFCB$. The actual current in any conductor $BC$ is the difference $x - y$ of the mesh currents in the two meshes it separates.

We then do not need to consider Kirchoff Law I. (§ 114); for the sum of currents entering each point

vanishes necessarily. Thus at $C$ the currents in $BC$, $FC$, $DC$ are $x - y$, $y - z$, $z - x$; whose sum is zero. Kirchoff Law II. is then applied to each independent mesh.

**116. The Wheatstone Net.**—Consider six resistances $F$, $G$, $H$, $L$, $M$, $N$ arranged as in Fig. 49. Let there be an electromotive force $E$ in branch $L$. Insert mesh currents $x$, $y$, $z$; so that evidently the current in $L$ is $x$; in $H$ is $x - y$; etc.

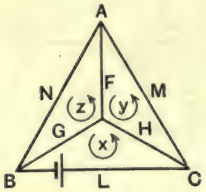

Fig. 49.

Apply Kirchoff's Law II. to mesh $LGH$; we have

$$Lx + H(x - y) + G(x - z) = E$$

or

$$(L + H + G)\,x - Hy - Gz = E \quad (1)$$

Similarly the equations for the other meshes are

$$-Hx + (M + F + H)\,y - Fz = 0 \quad (2)$$

$$-Gx - Fy + (N + G + F)\,z = 0 \quad (3)$$

Before giving the general solution, we consider some very useful results which follow if the resistances be related in a particular manner.

**117. To Find the Resistance of a Wire by the "Bridge" Method.**—In § 116 eliminate $x$ from (2) and (3).

$$\{MG + GH + HF + FG\}\,y = \{NH + GH + HF + FG\}\,z.$$

If such an adjustment is made that

$$MG = NH, \text{ then } y = z,$$

$$\therefore\ y - z = 0,$$

and there is no current in the conductor $F$. This is the principle employed in the "Bridge" method of determining the resistance of a wire. The wire considered is connected as resistance $M$ in figure. $F$ is a galvanometer, and the values of $N$, $G$, $H$ are adjusted until there is no

deflection of the galvanometer. Then the current in $F$ is zero, or

$$y - z = 0,$$
$$\therefore\ MG = NH, \text{ or } M/N = H/G.$$

This gives the resistance $M$ if $N$, $G$, $H$ are known.

**118. To Find the Resistance of a Galvanometer by Lord Kelvin's Method.**—If $MG = NH$ as above, and therefore there is no current in $F$, it cannot make any difference to the currents in the other conductors if we make $F$ infinite by opening a key in $F$.

We can prove this analytically. Since $y = z$, equations (1) and (2) become

$$(L + H + G)x - (H + G)y = E$$
and
$$-Hx + (M + H)y = 0.$$

As these equations do not contain $F$, the values of $x$ and $y$ are independent of $F$. In fact

$$x = E(M + H)/(ML + MH + MG + LH)$$
$$y = z = EH/(ML + MH + MG + LH)$$
$$x - y = x - z = EM/(ML + MH + MG + LH).$$

Let $M$ be the required resistance of a galvanometer. Insert a key in $F$ and adjust the resistances $N$, $G$, $H$ until the current indicated by the galvanometer does not alter whether the key be open or closed. When this is the case, by above, $MG = NH$ or $M/N = H/G$, hence $M$ is found if $N$, $G$, $H$ be known.

**119. To Find the Resistance of a Cell by Mance's Method.**—Let $z$ be eliminated from (1) and (2). Then

$$(FL + GH + HF + FG)x - (GM + GH + HF + FG)y = EG.$$

If such an adjustment is made that $FL = GM$, then

$$x - y = EG/(FL + GH + HF + FG),$$

and the solution does not involve the value of $N$. But $x - y$ is the current in $H$, therefore the current in $H$ is unaltered by opening a key in $N$ provided $FL = GM$.

**Example.**—Let $L$ be the required resistance of a cell. Put a key in $N$ and a galvanometer in $H$, and adjust the other resistances until the galvanometer deflection is unaltered by opening or closing the key. Then $L$ is given by the equation $LF = MG$.

**120. Full Solution of the Equations for the Wheatstone Net.**—Eliminating $x$ from (2) and (3)

$$(MG + GH + HF + FG)y = (NH + GH + HF + FG)z.$$

Eliminate $y$ from (2) and (3)

$$(MG + GH + HF + FG)x = (FM + FN + FH + FG + MN + MG + NH + GH)z.$$

Hence we can write

$$\Delta\, x = FM + FN + FH + FG + MN + MG + NH + GH,$$
$$\Delta\, y = NH + GH + HF + FG,$$
$$\Delta\, z = MG + GH + HF + FG,$$

where $\Delta$ is to be determined by the equation (1).

If we adopt the condensed notation $(FGH)$ for the sum of products of the resistances in the sides of triangle, $FGH$, two at a time, we get the forms

$$\Delta\, x/E = MG + NH + (FMN) + (FGH),$$
$$\Delta\, y/E = NH + (FGH),$$
$$\Delta\, z/E = MG + (FGH),$$
$$\Delta\, (y - z)/E = NH - MG,$$
$$\Delta\, (x - z)/E = NH + (FMN),$$
$$\Delta\, (x - y)/E = MG + (FMN).$$

Now

$$Lx + H(x - y) + G(x - z) = E,$$
$$\therefore E\Delta = L(FM + FN + FH + FG + MN + MG + NH + GH) + H(FM + FN + MN + MG), + G(FM + FN + MN + NH).$$

This is a symmetrical expression of 16 products, each of 3 factors. All the products of the six resistances, three together, are represented except the four formed by multiplying together the resistances in the three sides of a triangle,

$$LGH + MHF + NFG + LMN.$$

**121. The Metre Bridge.**—This consists of three brass pieces $A$, $B$, $D$ whose electrical resistance is so small that they can be regarded as points, and correspond to the points $A$, $B$, $D$ in Fig. 50. $A$ and $D$ are connected by a uniform wire whose length is generally a metre. The whole is mounted on a non-conducting board, and there are terminals, as indicated, so that the galvanometer can be connected from $A$ to $D$, one pole of the battery connected to $B$, and resistances $R$ and $S$ inserted in gaps $AB$ and $BD$. The other pole of the battery is applied at various points $C$ along the wire $AD$ until the galvanometer shows no deflection. Let $P$, $Q$ be resistances of $AC$, $CD$: then evidently $P$, $Q$, $R$, $S$ correspond to our previous $M$, $H$, $N$, $G$; and the condition $GM = HN$ gives $PS = QR$ or $P : Q = R : S$. Of course $P/Q = AC/CD$, as the wire is uniform.

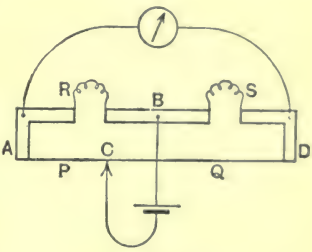

Fig. 50.

**122. The Post Office Box.**—This is a resistance box with three sets of resistances. Its terminals are marked with the conventional letters $G$, $B$, $L$, $E$, often placed in the boxes. To distinguish the two $G$'s and $B$'s we shall have to use the double letters $GL$, $EB$, etc., as names of single points. The resistance $P$ between $G$ and $B$, and also $Q$ between $B$ and $GL$, can usually be 10, 100 or 1000 as desired; thus the ratio $P/Q$ can be 1, 10, 100, 1/10 or 1/100. $R$ can be any integer from 1 to 10,000. The resistance $S$ to be found is connected from $L$ to $E$; the galvanometer from $G$ to $GL$, the battery from $B$ to $EB$. Instead of $B$, $G$ one often uses the

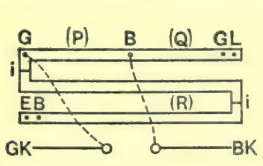

Fig. 51.

terminals $BK$, $GK$ on two keys which connect when pressed with $B$ and $K$. For a balance, $P/Q = R/S$; $\therefore S = R \times 1$, 10, 100, 1/10 or 1/100.

**123. The Carey Foster Bridge.**—It has four gaps as shown. Its object is to find the difference between two resistances $P$ and $Q$ very nearly equal. $R$ and $S$ are fixed resistances nearly equal. Let $r$ = resistance of unit length of bridge wire.

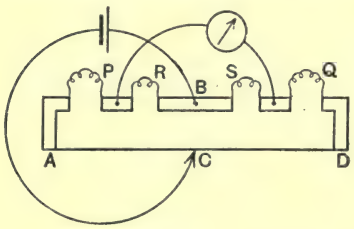

Fig. 52.

When a balance is obtained at $C$,

$$\frac{P + r.\ AC}{Q + r.\ CD} = \frac{R}{S}$$

$$\therefore \frac{P + r.\ AC}{P + Q + r.\ AD} = \frac{R}{R + S}$$

Interchange $P$ and $Q$; get a balance at some point $C'$. Then

$$\frac{Q + r.\ AC'}{P + Q + r.\ AD} = \frac{R}{R + S}$$

Comparing these equations

$$P + r.\ AC = Q + r.\ AC'$$
$$\therefore\ P - Q = r\,(AC' - AC) = r.\ CC'$$

(taking due note of sign). Hence $P - Q$ is found if $r$ be known. One can determine $r$ by use of two resistances whose difference is accurately known: or as in Ex. 9 below.

**124. Extension of the Carey Foster Method.**—Let a resistance $T$ be placed between $A$ and $D$ in Fig. 52. . . . When a balance is obtained, there is no current in the galvanometer $G$; and if the other mesh currents be $x$ in mesh formed by $P$, $R$, $B$

and $AC$; $y$ in $B$, $S$, $Q$ and $CD$; $z$ in $ACD$ and $T$; we have, $B$ being the battery resistance,

$$0 = Rx + Sy \quad (1)$$
$$E = B(x - y) + (R + P)x + r.AC(x - z) \quad (2)$$
$$-E = B(y - x) + (S + Q)y + r.CD(y - z) \quad (3)$$
$$0 = r.AC(z - x) + r.CD(z - y) + Tz. \quad (4)$$

Equation (4) gives $(T + r.AD)z = r.AC.x + r.CD.y$.

Add (2) and (3), $O = (R + P)x + (S + Q)y - r.AD.z + r.AC.x + r.CD.y$.

Eliminate $z$,

$$O = x\{R + P + Tr.AC/(T + rAD)\} + y\{S + Q + Tr.CD(T + r.AD)\}.$$

But by (1) $x/y = -S/R$.

$$\therefore \frac{R + P + r.T.AC/(T + rAD)}{R} = \frac{S + Q + r.T.CD/(T + r.AD)}{S}$$

$\therefore$ If $r' = r.T/(T + r.AD)$

$$\frac{P + r'.AC}{Q + r'.CD} = \frac{R}{S},$$

which is the same as in § 123, with $r'$ substituted for $r$. Evidently $r' < r$. So the effect of inserting $T$ is that the apparent resistance per unit length of the bridge wire is altered quite arbitrarily, if only less than the actual resistance per unit length. One usually adjusts so that $r'$ = a convenient submultiple of an ohm.

**Ex. 1.**—In a common meter bridge wire a standard ohm is placed at $R$, and a fine copper wire, whose temperature can be varied, at $S$. When $S$ is at freezing point, a balance is obtained at $C$ where $AC$ = 42 cm. When $S$ is at boiling point, the balance is at 34·4 cm. Find the resistances at these temperatures, and the temperature coefficient of the resistance.

Let $S_0$, $S_{100}$ be the resistances at 0° and 100°.

Evidently $S_0/1 = (100 - 42)/42 = 1·380$.

Also $S_{100} = (100 - 34·4)/34·4 = 1·907$.

The change of resistance = ·527,

Hence the change per 1° = ·00527.

The temperature coefficient = change per 1° ÷ resistance at freezing point = ·00527/1·380 = ·00382.

**Ex. 2.**—In a Carey Foster bridge $P$ is a standard ohm and $Q$ an approximate ohm. The balances when $P$ and $Q$ are as in figure and reversed are respectively at 56·7 and 49·2. Another experiment is made in which $P$ is used in parallel with a 20 ohm coil, so

that the joint resistance is 20/21. The balances are now at 62·7 and 43·3. Find the resistance $r$ per cm. of bridge wire, and the exact resistance of $Q$.

| | |
|---|---|
| By theory | $1 + 56{\cdot}7r = Q + 49{\cdot}2r,\ Q = 1 + 7{\cdot}5r.$ |
| Also | $20/21 + 62{\cdot}7r = Q + 43{\cdot}3r,\ Q = 20/21 + 19{\cdot}4r$ |
| Hence | $1/21 = 11{\cdot}9r, \quad \therefore\ r = {\cdot}0040.$ |
| And | $Q = 1{\cdot}030.$ |

**Ex. 3.**—Suppose the above experiment (Ex. 2) carried on when $Q$ is at temperature 15°. When $Q$ is at 100°, we get balances, with the 1 ohm coil $P$ as above, at 58·8 and 43·3. Find the temperature coefficient of $Q$.

At 100°

$$Q_{100} + 43{\cdot}3r = 1 + 58{\cdot}8r$$
$$Q_{100} = 1 + 15{\cdot}5 \times {\cdot}004 = 1{\cdot}062.$$

But by previous example $Q_{15} = 1{\cdot}030$

$\therefore$ change for 85° = ·032.

Hence by proportion the change for 1° is ·032/85 and for 15° is ·006. We infer the resistance at freezing point = 1·024.

Hence the temperature coefficient is

$$·032 \div \{85 \times 1{\cdot}024\} = {\cdot}00037.$$

**Ex. 4.**—A column of mercury in a tube of diameter ·5 mm. and length 100 cm. is placed in one arm of a metre bridge; and a 1 ohm standard coil in the other. A balance is obtained at 17 cm. from the end nearest to the 1 ohm. Find the resistance of the mercury column and the specific resistance of mercury at the temperature of the experiment.

**125. Disturbance of Balance of Wheatstone Net.**—By § 120 the current in the galvanometer is

$$y - z = E(NH - MG)/\Delta.$$

There is a balance if the current is zero, and $\therefore$ if $NH = MG$. If one of the four resistances $N$, $M$, $G$, $H$ be slightly altered, this can be expressed by putting

$$N = MG/H + n.$$

Then

$$y - z = EHn/\Delta,$$

where, of course, $\Delta$ is nearly constant when the charge of $N$ is not large. Hence the current is proportional to

the difference $n$ between the actual resistance and the balancing resistance, provided $n$ is not large.

By similar work, if $M$ differ by $m$ from the value necessary to balance,

$$y - z = -EGm/\Delta .$$

And changes $m$, $n$ produce the same current if

$$nH = -mG ;$$

But $NH = MG$, hence

$$m/M = -n/N.$$

**Example.**—A P.O. box is used to measure a small resistance. Hence, using the notation of § 122, $S$ is small, and we arrange $P/Q = 100$ so that $S = R/100$. No exact balance is found, but the galvanometer deflection (assumed proportional to current) is 13 divisions to right when $S = 27$ and 10 to left for 28. Find the correct value of $R$ for a balance.

Evidently $R - 27 : 28 - R = 10 : 13$

$$R = 26{\cdot}4$$

and $S = {\cdot}264$ ohm.

**126. The Potentiometer or Poggendorff Net.**—A diagrammatic view is given. $E$ and $e$ represent the E.M.F.s of the corresponding batteries. $P$, $Q$, $G$ are the resistances of $AC$, $CDBA$, $AGC$. The mesh currents are $x$ and $y$ in $ABCA$, $ACGA$. The equations for the two meshes are

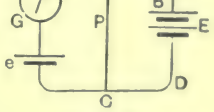

Fig. 53.

$$E = P(x - y) + Q x ;$$
$$-e = P(y - x) + G y.$$

The most useful case is when we adjust for zero current in the galvanometer in arm $G$. Then $y = 0$.

$$\therefore\ E = (P + Q) x, \quad e = Px ;$$
$$\therefore\ e = PE/(P + Q).$$

The experiment is usually to compare E.M.F.'s. Then $E$ is a strong constant battery, and the two batteries $e$ and $e$ to be compared are connected to the same terminal of the

galvanometer. A balance is obtained first for $e$ with resistances $P$ and $Q$; then for $e'$ with $P'$ and $Q'$.

If $P + Q$ is kept constant, $= P' + Q' = R$, then

$$e = PE/R,\ e' = P'E/R$$

and

$$e/e' = P/P'.$$

The battery $e$ can be a constant battery (*e.g.* a Clark cell) kept only for comparisons of E.M.F. Since no current goes through it in the required position, and only small transient currents pass as the position is sought, the cell is not liable to get out of order, as it would do if used for the Bosscha method (§ 131).

**127. Forms of Potentiometer.**—In one form of potentiometer, $AD$ is a bridge wire, either straight as in the ordinary meter bridge or bent to secure greater length. In another, $AC$ and $CD$ are two ordinary boxes each capable of giving resistances (say) from 1 ohm to 100 ohms. As the resistance of one is increased that of the other is diminished, their sum being kept constant and $= 100$. So that every plug withdrawn from either box has its corresponding plug in the other box inserted.

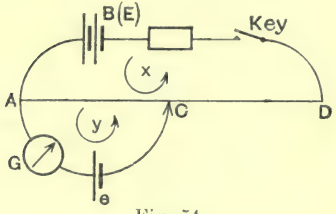

Fig. 54.

A third form of potentiometer has two resistance boxes used as above, and a bridge wire of resistance 1 ohm between them. The total resistance $P + Q$ is then 101 ohms: and the value of $P$ is given by one box and part of the bridge wire: the box giving a whole number of ohms and the wire a decimal.

**128. Other uses of the Potentiometer.**—In the case of a balance, we have seen $e = Px$ (§ 126). If the cell is a standard one, the value of $e$ is accurately known. Since

$P$ is also known, the potentiometer gives instantly a method of measuring currents. This method is often much more reliable than the measurement by a direct reading galvanometer. It is very rare that a galvanometer can be relied on so certainly as a standard cell or a resistance.

**Ex. 1.**—A potentiometer wire has a length of 400 cm., or other arbitrary units. A balance is obtained with a Clark cell at reading $AC = 257$ and with a Dry cell at 280. Find the E.M.F. of the Dry cell (that of the Clark = 1·433).

$$\text{The E.M.F.} = 1{\cdot}433 \times 280/257 = 1{\cdot}56.$$

**Ex. 2.**—A balance (with the same potentiometer but different current) is found for a Bunsen cell at 340 and a Daniell at 205. Find the E.M.F. of the Bunsen, that of the Daniell being 1·08. Find also the Potential Difference between the ends of a wire.

$$\text{Bunsen} = 340 \times 1{\cdot}08/205 = 1{\cdot}79 \text{ volts.}$$

$$\text{P.D.} = 400 \times 1{\cdot}08/205 = 2{\cdot}17 \text{ volts}$$

**Ex. 3.**—A potentiometer is made up with two resistance boxes as described in § 127; the sum of the resistances being 100. A Clark cell is balanced when the resistance $AC$ is 65 ohms. Find the current $x$.

$$e = Px, \quad \therefore 1{\cdot}433 = 65\,x;$$

$$x = {\cdot}022 \text{ ampere.}$$

**Ex. 4.**—A potentiometer net is set up in which $P$ and $Q$ are two boxes whose sum is not kept constant. There is a galvanometer which is to be standardised in the arm $Q$, and $e$ is a Daniell (1·08). $P$ is adjusted so as to produce a balance in a sensitive galvanometer at $G$. The galvanometer indicates 20·4, 31, 41·6, 63·2, 97 milliamps when $P = 54, 36, 27, 18, 12$ ohms.

Tabulate the errors of the galvanometer.

Since $x = e/P$, the true currents in amperes are the quotients of 1·08 by the given resistances; and therefore are 20, 30, 40, 60, 90 milliamps. The errors (negative) are ·4, 1, 1·6, 3·2, 7: and should be plotted against the galvanometer readings.

**Ex. 5.**—An electrometer is connected to two points 20 cm. apart on a uniform potentiometer wire. The current is varied so that the scale readings are 10, 20, 30, 40, 50. A Clark cell is balanced correspondingly at distance 276, 133, 85, 62, 48 cm. from the electrometer.

Clearly in the first case 276 cm. corresponds to a P.D. of 1·433.

hence 20 cm. corresponds to $20 \times 1{\cdot}433/276 = {\cdot}104$. Hence 10 scale divisions indicate 104 millivolts.

| | | | | |
|---|---|---|---|---|
| Similarly | 20 | scale divisions | indicate | 215 |
| | 30 | ,, | ,, | 337 |
| | 40 | ,, | ,, | 462 |
| | 50 | ,, | ,, | 597 |

**129. Rectangular Array of Cells.**—Let there be $m$ sets in parallel, each containing $n$ similar cells in series. In figure, $m = 4$ and $n = 5$. Let $E$, $B$ be the electromotive force and resistance of each cell. Let $R$ be the external resistance. If $x$ be the current through every cell, $mx$ must be the current through the resistance $R$.

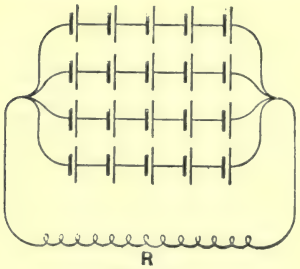

Fig. 55.

Applying Kirchoff's Law to one mesh,

$$nBx + Rmx = nE,$$
$$\therefore\ x = nE/(mR + nB).$$

And the current $C$ in $R$ is given by

$$C = mx = mnE/(mR + nB).$$

To find the heat developed per second. In each of the $mn$ cells it is $Bx^2$ ; in the outside wire it is $R\ (mx)^2$. Therefore the total is

$$\begin{aligned} H &= mnBx^2 + Rm^2x^2, \\ &= mx^2(nB + Rm), \\ &= mnxE\,; \end{aligned}$$

so that it is equivalent to the energy supplied per second, as it ought to be. It also

$$= mn^2E^2/(mR + nB).$$

The zinc consumed per second is evidently $mnxz$ if $z$ be the electrochemical equivalent of zinc. It is therefore proportional to the heat supplied.

**130. Maximum Current with given number of Cells.**—Let $N$ be the number of cells; put $N = mn$. Then the current

$$C = mnE/(mR + nB) = NE/(mR + nB).$$

Now $C$ is greatest (given $N$) when the denominator $mR + nB$ is least. But the product of these terms is $NRB$, which is constant. And the sum of two otherwise unrestricted positive quantities whose product is constant is least if the quantities are equal. Hence $C$ is greater the more nearly $mR = nB$. Analytically, this is proved by the identity

$$\begin{aligned}(mR + nB)^2 &= (mR - nB)^2 + 4mnRB,\\ &= (mR - nB)^2 + \text{constant}.\end{aligned}$$

Hence, if possible,

$$\begin{aligned}mR &= nB, \text{ and } mn = N,\\ \therefore\ m^2 &= NB/R.\end{aligned}$$

But since $m$ is restricted to be an integral factor of $N$, it cannot always equal $\sqrt{NB/R}$, and we are obliged to take the two nearest factors, one on each side of $\sqrt{NB/R}$, and find which of them makes $mR - nB$ numerically smallest.

**Example.**—120 cells of resistance 1 ohm are to be grouped to send a maximum current through a wire of 20 ohms resistance. Find the best method.

Here $mn = 120$, and $20m - n$; hence $m^2 = 6$, and the nearest factors to $\sqrt{6}$ are 2 and 3.

Hence $m = 2$ or 3, $n = 60$ or 40, $20m - n = -20$ or $+20$. It happens therefore that the two arrangements, two series each of 60 and three series each of 40, give the same current in $R$.

The second arrangement is preferable; it uses less zinc.

**131. Bosscha or Lumsden Net.**—This is to compare the electromotive force of two active* cells, while the potentiometer method is to compare electromotive forces of cells through which a current is not passing. The

*The method should not be used for a Clark or other standard cell.

cells of E.M.F. $E_1$ and $E_2$ and resistance $B_1$ and $B_2$ are arranged with resistance boxes $R_1$ and $R_2$ as in figure.

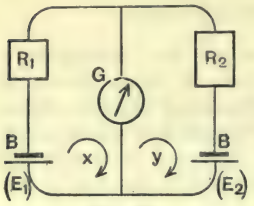

Fig. 56.

Let $x$ and $y$ be mesh currents, and $G$ the resistance of galvanometer. The equations are

$$(R_1 + B_1)x + G(x - y) = E_1,$$
$$(R_2 + B_2)y + G(y - x) = E_2.$$

If there be no current in galvanometer $G$, $x = y$,

$$\therefore E_1 : E_2 = R_1 + B_1 : R_2 + B_2.$$

If another balancing adjustment be obtained with different resistances,

$$E_1 : E_2 = R_1' + B_1 : R'_2 + B_2,$$
$$\therefore \text{ finally } E_1 : E_2 = R_1' - R_1 : R'_2 - R_2.$$

If there be no balance, solving for the current in the galvanometer we obtain

$$x - y = \left(\frac{E_1}{R_1 + B_1} - \frac{E_2}{R_2 + B_2}\right) \Big/ \left(1 + \frac{G}{R_1 + B_1} + \frac{G}{R_2 + B_2}\right).$$

The potential difference at the points $A$ and $B$ is of course

$$= V = G(x - y) = \left(\frac{E_1}{R_1 + B_1} - \frac{E_2}{R_2 + B_2}\right) \Big/ \left(\frac{1}{G} + \frac{1}{R_1 + B_1} + \frac{1}{R_2 + B_2}\right)$$

and the potential difference which would exist if the galvanometer were removed or replaced by an electrometer ($G = \infty$ , $1/G = 0$)

$$= \left(\frac{E_1}{R_1 + B_1} - \frac{E_2}{R_2 + B_2}\right) \Big/ \left(\frac{1}{R_1 + B_1} + \frac{1}{R_2 + B_2}\right).$$

The student can calculate $x$ and $y$ separately as an exercise. In the balancing adjustment

$$x = y = (E_1 + E_2)/(R_1 + B_1 + R_2 + B_2);$$

and the assumption is made that the $E$'s do not alter with the current.

**Ex. 1.**—Three Daniells of resistance 2 ohms and E.M.F. 1 volt are in series with a resistance of 4 ohms. Find the P.D. between the two terminals of one Daniell cell.

We shall do this a shorter way, applicable when there is only one current. The E.M.F. = 3 and resistance = 7, therefore current = 3/7. If $V$ be P.D. required of $A$ over $B$, 1 = E.M.F. between $A$ and $B$, and 1 = resistance.

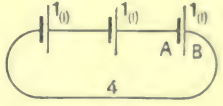

Fig. 57.

$$\therefore \quad V + 1 = 1 \times 3/7 \text{ (§ 113)},$$
$$\therefore \quad V = -4/7.$$

That is, the P.D. of $B$ over $A$ is 4/7 volts.

**Ex. 2.**—If $A$ were connected to $B$ by a galvanometer of resistance 2 ohms, what currents would flow.

Let $x$ be mesh current through the one cell $AB$ and $y$ through the other two cells ; $x - y$ through galvanometer. Then

$$x + 2(x - y) = 1,$$
$$6y + y - x = 2,$$
$$\therefore \ x = 11/19,\ y = 7/19,\ x - y = 4/19.$$

The P.D. of $A$ and $B = 2(x - y) = 8/19$.

**Ex. 3.**—Two cells are compared by Lumsden's method. A balance is obtained with resistances 50 and 125, and also 100 and 212. Find ratio of E.M.F.'s, and show that one cell has a resistance exceeding 20.

$$\frac{E_1}{E_2} = \frac{50 + B_1}{125 + B_2} = \frac{100 - 50}{212 - 125} = \frac{50}{87}$$

Hence $$4350 + 87B_1 = 6250 + 50B_2.$$

As $B_2$ cannot be negative, $B_1 > 1900/87 > 20$.

**Ex. 4.**—A Daniell (E.M.F. 1·08) is compared with a Bichromate by Lumsden's method. The resistances with the Daniell and Bichromate were first 6 and 14 ; afterwards 20 and 35. Find the E.M.F. of the Bichromate, and the least possible resistance of the Daniell.

## 132. Examples illustrating Miscellaneous Experiments.

**Ex. 1.**—A very small resistance is to be found, between points $B$ and $C$ on a bar $ABCD$. The connections are made as drawn. The box at $A$ is a standard ohm and 2000 is 2000 ohms. $K$ is an

adjustable resistance. If $A$ be united to the terminal of the galvanometer we get a balance when $R = 59$. If $B$ be united, $R = 37$. If $C$ be united, $R = 15$. Find resistance $BC$.

By principle of Wheatstone's bridge,

$$1/AD = 2000/59$$
$$(1 + AB)/BD = 2000/37$$
$$(1 + AC)/CD = 2000/15$$

Hence
$$1/(1 + AD) = 2000/2059$$
$$1 + AD = 2059/2000$$

Also
$$(1 + AB)/(1 + AD) = 2000/2037$$
$$(1 + AC)/(1 + AD) = 2000/2015$$
$$\therefore\ BC/(1 + AD) = 2000/2015 - 2000/2037.$$

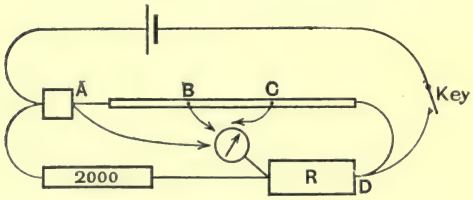


Fig. 58.

Hence finally

$$BC = \left(\frac{1}{2015} - \frac{1}{2037}\right) \Big/ \frac{1}{2059}$$
$$= \frac{2059}{2015} - \frac{2059}{2037}$$
$$= \cdot 0110.$$

**Ex. 2.**—A battery of resistance $B$ and a galvanometer of resistance $G$ are first connected in series with a resistance $R$; then $G$ is put in parallel with a resistance $S$ and the two are connected to the battery. The current in galvanometer is same in both cases; find relation between resistances.

In first case the total resistance is $R + B + G$, therefore current $= E/(R + B + G)$. In second case, if mesh currents be $x$ and $y$, as in figure 59,

$$E = Bx + S(x - y)$$
$$0 = Gy + S(y - x)$$
$$\therefore\ y = ES/(SG + SB + BG).$$

Hence $E/(R + B + G) = ES/(SG + SB + BG)$

$$\therefore\ BG = RS.$$

This arrangement can be used to find battery resistance $B$, if $G$ be known. If we connect a battery of small resistance to a box $B$, the combination can be regarded as a battery of resistance $B$. The above relation can then be used to find galvanometer resistance $G$ or (if this be known) a large resistance $R$ or a small one $S$.

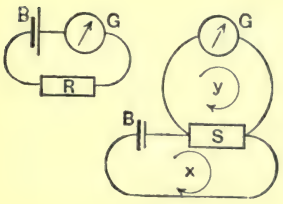


Fig. 59.

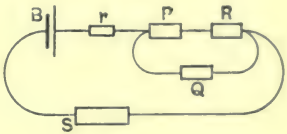


Fig. 60.

**Ex. 3.**—In arrangement in Fig. 60, the letters being resistances, find the condition that the P. D. at terminals of $r$ and of $P$ are equal.

Let $x, y, x + y$ be currents in $P, Q, r$. Then

$$x\,(P + R) - yQ = o,\ y = x(P + R)/Q.$$

The potential differences are $r\,(x + y)$, $Px$; and these are equal if

$$r = P/(1 + y/x) = PQ/(P + Q + R).$$

This method is used to find a very small resistance $r$. Usually $P$ and $Q$ are standard ohm coils,

$$\therefore\ r = 1\,(R + 2).$$

**Ex. 4.**—A battery (of negligible resistance) is connected to a voltameter and releases 30 c.c. of hydrogen in 5 minutes. A 5 ohm resistance is inserted, and the rate of production of hydrogen falls to 21 cm. per 5 minutes. Find the resistance of the voltameter, and the production of hydrogen if 10 ohms were inserted.

**Ex. 5.**—A battery, whose resistance is made up to 10 ohms by a resistance box, is put in series with a galvanometer of 100 ohms resistance, shunted with a 1/10 ohm coil. The same current is produced when the battery is connected direct to the galvanometer (unshunted) through a large resistance $R$. Calculate $R$.

**133. Resistance and Temperature.**—We have defined temperature coefficient (§ 41, end) and shown how to measure it (§ 53, Exs. 8, 10).

Strictly, the change of resistance with temperature is not quite uniform. If $t$ be the centigrade temperature, $R$ the resistance, and $A$ its value at freezing point,

$$R = A(1 + at + bt^2).$$

The constants $a$ and $b$ depend somewhat on the physical state of the wire as well as on its material.

As approximate values $a = \cdot00367$, $b = \cdot00000038$ for platinum.

The student may verify that, with these values of $a$ and $b$, $R$ is almost zero at the temperature $-273°$ C.

For various very pure metals Matthiessen has given the mean values,

$$a = \cdot003824, b = + \cdot00000126.$$

For $t = -273$ these would give $R = \cdot006A$, or the resistance is 1/167 of that at freezing point. It may be stated generally that even bodies generally regarded as insulators become good conductors when cooled to the very low temperatures like that of liquid air, and it is not impossible that all bodies without exception would conduct at the thermodynamic absolute zero.

**134. The Platinum Resistance Thermometer.**—A platinum thermometer is designed to measure temperatures beyond the range of a mercury thermometer, and also to measure ordinary temperatures and temperature differences with a precision unattainable by mercury or air thermometers. A form devised by Callendar and Griffiths is shown diagrammatically (Fig. 61). $ab$ is a fine platinum wire, wound on a mica frame and enclosed in a bulb (glass or porcelain) to be exposed to the temperatures measured. $Aa$ and $Bb$ are the thick wires (platinum preferably), called leads, which conduct to it. $CD$ is another wire of same material whose resistance is the sum of the resistances of $Aa$ and $Bb$. If the difference of resistance of

*AabB* and of *CD* be found, this is the correct resistance of *ab* at the temperature of the enclosure.

For accurate work another identical thermometer is taken, whose corresponding terminals can be indicated by dashed letters. It is at temperature $\theta'$ say, while the first one is at $\theta$. The parts $AB$ and $C'D'$ are part in series, also the parts $CD$ and $A'B'$; and therefore the difference of resistances $ABC'D'$ and $CDA'B'$ is precisely that of the fine wires $ab$, $a'b'$. It is practically proportional to the temperature difference $\theta - \theta'$. Supposing $ab$ to be about 1 ohm, the difference of resistance for $\theta - \theta' = 1°$ will be about ·00367 ohm. It is quite easy to measure by the Carey Foster bridge to a much smaller quantity.

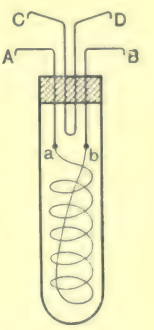

Fig. 61.

**135. Simplified Formulae.**—If only one platinum wire is used, the temperature is inferred from the resistance. We have

$$R = A(1 + at + bt^2).$$

If $t = o$, $R = A$; hence $A$ is the resistance at freezing point. Let $B$ be the resistance at boiling point, so that

$$B = A\{1 + 100a + 10000b\}$$

$$\therefore \frac{R - A}{B - A} = \frac{t(a + bt)}{100(a + b.100)}.$$

If $b$ were $o$, it is evident that $t = 100(R - A) \div (B - A)$. This quantity is denoted by $p$, and is called the "platinum temperature." It is the temperature calculated on the erroneous hypothesis that the temperature coefficient of platinum is constant.

Hence

$$\frac{p}{t} = \frac{a + bt}{a + b.100},$$

$$\therefore \frac{p - t}{t} = \frac{b(t - 100)}{a + b.100}.$$

Different observers give very different values for the

temperature coefficient of platinum. They range from $a = \cdot0024$ to $\cdot0038$. The divergence is undoubtedly due to the different physical state of the samples of platinum examined.

If we use

$$a = \cdot0036 = 3\cdot6 \times 10^{-3}, \quad b = -5\cdot4 \times 10^{-7},$$

then

$$b/a = -1\cdot5 \times 10^{-4},$$

hence the above formula gives

$$\frac{p-t}{t} = -1\cdot5 \times 10^{-4} \times (t - 100), \text{ nearly};$$

or

$$t - p = 1\cdot5 \frac{t}{100}\left\{\frac{t}{100} - 1\right\},$$

and approximately

$$= 1\cdot5 \frac{p}{100}\left(\frac{p}{100} - 1\right).$$

This formula is recommended by Callendar. It seems to be nearly correct for samples of platinum whose coefficient $a$ has very different values, whence $a/b$ must be nearly constant.

**Example.**—If $A = 1\cdot20$, $B = 1\cdot50$, find the temperature indicated by a resistance $1\cdot92$.

Evidently $p = 100(R - A)/(B - A) = 100 \times \cdot72/\cdot30 = 240$.

Hence $t - p = 1\cdot5 \times 2\cdot4 \times 1\cdot4 = 5°$, $\therefore\ t = 245°$ C.

# CHAPTER VIII.

## THE MAGNETIC FIELD.

**136. Magnetic Matter. Point Pole.**—Many magnetic phenomena have at first sight a close resemblance to electric phenomena. We shall find, however, fundamental differences. The elementary definitions can be adopted from electrostatics with very slight changes.

We speak of two kinds of magnetic matter or magnetism, North magnetism and South magnetism. North magnetism is the magnetism observed on that end of a freely suspended compass needle which points nearly to the north in England, and indeed in most countries.

The quantity, in a given space, of either kind of magnetic matter will generally be taken to mean the excess of the quantity of that kind over the opposite kind.

**A Point Pole** is a body containing magnetic matter, and so small that its dimensions may be disregarded. Nothing quite resembling a point pole exists in nature. The nearest resemblance is one end of a uniformly magnetised steel rod. This pole is rigidly connected with another point pole of the contrary kind at the other end of the rod. We shall be obliged at first, however, to speak of detached point poles as though such things were possible.

**137. Coulomb's Law.** (Cf. § 2.) The force exerted by one point pole on another is in the straight line joining them. It is proportional to the quantity of magnetic matter in each, and to the inverse square of the distance between them. It is a repulsion if the poles are

both $N$ or both $S$,* and an attraction if one is $N$ and the other $S$.

If $m_1$ and $m_2$ be the charges measured in any convenient units, and if $P$ be the force and $r$ the distance apart,

$$P \propto m_1 m_2 / r^2,$$

or

$$P = m_1 m_2 / \mu r^2,$$

$\mu$ being a constant.

**138. Unit Pole.**—The unit quantity of magnetism or the unit pole is such that, collected at a point, it would repel an equal and like quantity similarly collected at a point at unit distance (in air) with unit force. The accepted unit is that which repels a like equal pole at a distance of one centimetre with the force of one dyne. The surrounding medium is air in the definition, but the forces hardly depend on the medium at all, unless it be iron or one of a very few substances. It is sensibly the same if air be replaced by oil, vulcanite, brass, water, etc.

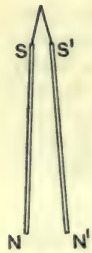

Fig. 62.

**Ex. 1.**—Two magnets 10 cm. long and of precisely equal strength are hung by cords side by side from a point. It is observed that their upper ends, to which the cords are fastened, are 1 cm. apart and the lower ends 2 cm. apart. If the weight of each magnet be 5 gm., determine the pole strength, and inclination of cords, neglecting all forces except the weights of the magnets, tensions of cords, and mutual repulsions.

Let $m$ be the pole strength and let $P$ be the mutual repulsion of lower poles, so that

$$P = m^2/2^2 = m^2/4.$$

To eliminate the tensions of the cords take moments about upper pole $S$ of magnet $SN$. Then, very nearly,

$$10\, m^2/4 = 5\, g \times 1/2,$$
$$\therefore\ m^2 = g = 981,$$
$$\therefore\ m = 31{\cdot}6.$$

---

*We shall use these letters as contractions for North and South, when these terms are used in their magnetic sense.

The repulsion between the lower poles is $m^2/4 = g/4$, that between the upper poles is $m^2/1 = g/1$, therefore the horizontal force acting on each magnet is $5\ g/4$. The vertical force (weight of magnet) $= 5g$; hence the tension of the cord must act so that its vertical component is four times its horizontal component; hence the slope of the cord to the vertical is $\tan^{-1} 1/4 = 14°$.

**Ex. 2.**—If the same magnets were suspended with their $S$ poles in contact, calculate how far apart their $N$ poles would hang.

**Ex. 3.**—If the same magnets be fixed $\frac{1}{2}$ cm. apart, and parallel, so that the line joining the $S$ pole of each magnet to the $N$ pole of the other is perpendicular to the line of each magnet, calculate the total attraction exerted by each on the other.

**139. Field.**—A magnetic field is a region of space within which the influence of magnetic bodies extends. The field of a particular body is the region in which its action is sensible with such means of observation as we possess.

**Field Strength** or **Force** or **Field Intensity** or **Field** simply) is the force that a unit $N$ point pole would experience at the place considered, if it exercised no influence on the magnetic distribution in its neighbourhood.

The mathematical consequences of these definitions are the same as those of the corresponding definitions in electrostatics, see §§ 8, 9. Field is a vector quantity and the field due to a pole $m$ at a distance $r$ from that pole has magnitude $m/r^2$, and is in the direction of $r$.

**140. Potential.**—The excess of potential at one point over another point in a magnetic field is the work which the magnetic forces would do on a unit $N$ pole travelling (in air) from the first point to the second point.

The mathematical consequences of this definition are the same as for Electric Potential, § 10. As in § 11, the magnetic potential due to a pole $m$ at a distance $r$ from it has magnitude $m/r$, and is a scalar quantity, so that the potentials due to different poles are simply added with due regard to sign.

All the electric and magnetic phenomena which depend

immediately on Coulomb's Law are so closely analogous that we shall assume identical mathematical formulae without further proof, and shall proceed to explain only properties in which marked differences appear. Practically the whole of Chapter I. applies (except § 17 and certain consequences, for though Maxwell Lines are used in magnetism, Faraday Tubes are not). Chapter II. does not apply, for nothing analogous to a conductor exists in magnetism. The relevance of Chapter III. is considered in Chapter XII.

**Ex. 1.**—Draw a series of equipotential lines for the poles $10\,N$ and $10\,S$ placed at $N$ and $S$, 10 cm. apart.

The working and the result are the same as for Exercise 3, § 9 in Electrostatics.

**Ex. 2.**—Two magnets of length 10 cm. and pole strength $\pm 20$ are placed parallel, with like poles pointing the same way, so that the four poles occupy the four corners of a square. Find the field and potential at the centre of the square, and at the point midway between the $N$ poles.

**141. The Earth's Local Field.**—Under ordinary circumstances, the field strength due to the earth is constant in magnitude and direction over a wide area. For example, the field at Greenwich for 1910 has magnitude

$$F = \cdot 476,$$

slope to horizon

$$\delta = 67^\circ,$$

and therefore its horizontal component

$$H = F \cos \delta = \cdot 186,$$

and vertical component

$$V = F \sin \delta = \cdot 438.$$

The vertical plane which contains the earth's field makes now in Greenwich an angle of nearly 16° [15° 45′ in 1910] to the west of north. All these data are liable to rapid fluctuations about their mean values for the year, and the mean values alter very slowly with the time.

**142. An Ideal Simple Magnet** is a mathematical conception. It consists of two equal and contrary poles, $N$ and $S$, at a distance apart. It is represented, with some accuracy, by a carefully magnetised knitting-needle or steel bar, the distance between the poles being taken as roughly equal to the length of the bar.* But we shall see that it is generally not important that an experimental magnet should approach the ideal character. Every magnetised body has the property that the algebraic sum of all its $N$ magnetism (the $S$ being taken negative) is zero. The Ideal Simple Magnet is the simplest arrangement which fulfils this condition. By dividing the $N$ magnetism of an ordinary magnet into a set of poles, and the $S$ magnetism into a set respectively equal to them, we decompose the magnet into a number of ideal simple ones.

**143. Magnet in a Uniform Field.**—Let $NS$ be an ideal simple magnet, and let $KN$ be the direction of the uniform field. Let $F$ be the field strength, $+ m$ the magnitudes of the poles. On $+ m$ at $N$ the force exerted is $mF$, on $- m$ at $S$ the force is $mF$ the contrary way. These equal and contrary forces form a couple $mF \times SK$, where $SK$ is the perpendicular let fall from $S$ on to $KN$. If $l$ = length $SN$ of magnet, $\theta$ = its slope to the direction $KN$ of field, this couple

$$= mlF \sin \theta.$$

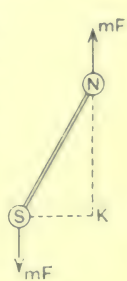

Fig. 63.

The product $ml$ is denoted by $M$, and is called the magnetic moment. Hence the couple

$$= MF \sin \theta.$$

This formula shows that the effect of a uniform field on an ideal simple magnet is identical with that on any other ideal simple magnet which has equal magnetic moment, and its axis in the same straight line or a parallel one.

* Robison's magnets, consisting of a narrow steel bar terminated by spherical steel knobs, are nearer to the ideal.

**Example.**—A magnet of length 10 cm. and pole strength 20 is placed horizontally, pointing magnetic east (perpendicular to the direction of $H$). Find the couple exerted on it (1) by the earth's horizontal field $H$, (2) by the vertical field $V$; taking the values from § 141.

**144. Compound Magnet.**—As in § 142, any compound magnet is replaceable by a number of ideal simple ones. Let their magnetic moments and axes be represented in magnitude and direction by the sides $AB$, $BC$, . . . $FG$ of a polygon taken in order. The first simple magnet has moment $AB$ and axis parallel to $AB$; it can therefore be replaced (so far as its magnetic moment is concerned) by poles $1\,S$ and $1\,N$, or $-1$ and $+1$, at $A$ and $B$, since these form a magnet of moment $AB \times 1 = AB$. Similarly the second is replaced by $-1$ and $+1$ at $B$ and $C$, and so on; the last being replaced by $-1$, $+1$ at $F$ and $G$.

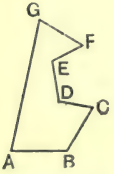

Fig. 64.

Thus at each point $B$, $C$ . . . $F$, we have $-1$ and $+1$ together, which cancel each other, and the whole system reduces to $-1$ and $+1$ at $A$ and $G$. The magnetic moment of this is $AG \times 1 = AG$.

If therefore we are dealing with phenomena which depend on magnetic moments only, we can compound together any system of simple magnets by the polygon law, or vector addition. That is, if a magnetic system can be decomposed into simple magnets whose moments are represented by the sides $AB$, $BC$, . . . $FG$ of a polygon taken in order, the system will act like a simple magnet whose moment is represented by the side $AG$ that closes the polygon.

This proposition can be called the **Polygon of Magnetic Moments**. The moment of the resultant simple magnet is called the Moment of the System.

**145. The Action of a Uniform Field on a Compound Magnet.**—Resolve the compound magnet into constituent single ones. Let their magnetic moments be represented by the sides of a polygon (§ 144), and let their resultant be $M$, the magnetic moment of the compound

magnet. The effect of the uniform field $F$ on each constituent simple magnet is given by § 143, and depends only on its magnetic moment. Hence (§ 144) the effect of $F$ on the whole magnet is the same as on the resultant simple magnet $M$.

If therefore $M$ makes an angle $\theta$ with the direction of $F$, the resultant effect is a couple of moment $MF \sin \theta$.

It is zero when $\theta = o$.

Its maximum value is $MF$, and takes place when $\theta = 90°$. This maximum becomes $M$ if $F = 1$. Hence we have a secondary or physical definition of Magnetic Moment.

The **Magnetic Moment** of any simple or compound magnet is the maximum couple which can be exerted on it by a uniform unit field. Its direction is that direction in the magnet which coincides with the field when no couple is exerted. The direction is sometimes called the Magnetic Axis.

**Ex. 1.**—Two magnets of moment $U$ and $V$ are fixed together so that their magnetic axes contain an angle $\alpha$. Find the magnetic moment of the combination, in magnitude and direction.

Let it be $W$, and let it make angles $\theta$, $\alpha - \theta$ with $U$ and $V$. Then

$$W^2 = U^2 + V^2 - 2UV \cos \alpha,$$

and

$$W/\sin \alpha = U/\sin (\alpha - \theta) = V/\sin \theta,$$

by Lami's Theorem for Forces.

**Ex. 2.**—Four magnets of moment 3, 5, 7, 9 are placed in order along the four sides of a square. Find the magnetic moment of the combination.

$$W = \sqrt{(3 - 7)^2 + (5 - 9)^2} = 4\sqrt{2}.$$

Its direction makes 45° with the sides, bisecting the angle between the magnetic axes 7 and 9.

**Ex. 3.**—Three coplanar magnets of moment 10, 20, 30, make equal angles of 120° with one another. Find the magnetic moment of the combination.

**146. Experimental Determination of Magnetic Moment, given H, the Earth's Horizontal Field.** The magnet is supported in such a way that its magnetic

axis is free to move in a horizontal plane. For instance, it may be suspended on a frictionless pivot or from a torsionless fibre.

The couple exerted on the magnet by $H$ is $MH \sin \theta$, if $\theta$ be the angle made by the axis with the direction of $H$, *i.e.* to the magnetic north. If now $K$ be the moment of inertia of the magnet about the axis of the suspension, the angular acceleration $= MH \sin \theta/K$. If $\theta$ be the small angle, this can be put equal to $MH\theta/K$. Hence the angular acceleration is (for small angles) proportional to the angular displacement, therefore the oscillation is simple harmonic, and its complete period is (by theory of simple harmonic motion)

$$T = 2\pi \sqrt{K/MH}.$$

For values of $K$ see table below.

**Moments of Inertia.**—In the following table $M$ is the mass of the body; $l$, $b$, $r$ are length, breadth, and radius; and $K$ is the moment of inertia taken about an axis through the centre of gravity, whose direction is perpendicular to the linear dimensions in the formula.

| Body. | K. |
|---|---|
| Thin uniform bar | $Ml^2/12$ |
| Bar of sensible breadth. Rectangular Parallelopiped | $M(l^2 + b^2)/12$ |
| Cylinder, about axis of cylinder | $Mr^2/4$ |
| Cylindrical bar, about axis perpendicular to axis of cylinder | $M\{l^2/12 + r^2/4\}$ |
| Hollow cylinder, external and internal radii $r$ and $s$ about axis perpendicular to its length | $M\{l^2/12 + (r^2 + s^2)/4\}$ |
| Hollow cylinder, about axis of cylinder | $M(r^2 + s^2)/2$ |
| Sphere | $M.2r^2/5$ |
| Ellipsoid; semiaxes $a$, $b$, $c$; about an axis perpendicular to $a$ and $b$ | $M(a^2 + b^2)/5$ |
| Hollow sphere, radii $r$ and $s$ | $\frac{2}{5}M(r^5 - s^5)/(r^3 - s^3)$ |

There is usually no sensible error in supposing the moment of inertia taken about a vertical axis through the centre of gravity. But if a correction be necessary for the fact that $G$ is not exactly on the axis, it can be found as follows :—If $K$ be the moment of inertia about any axis through the centre of gravity, the moment of inertia about an axis parallel to this axis, but at a distance $x$ from it, is $K + Mx^2$.

**147. Bifilar Suspension : Torsion Wire.**—Instead of using the time of oscillation we can use any means of measuring a couple. For instance, let a magnet be suspended horizontal at two points $C$ and $D$, from the two fixed points, $A$ and $B$, by two equal cords, $AC$ and $BD$. Let $AB = 2a$, $CD = 2b$, and $AB$ and $CD$ be horizontal, and $\phi$ be angle between them.

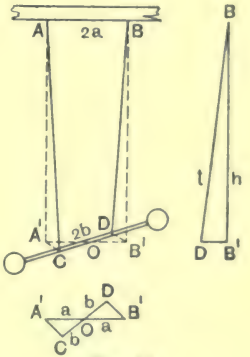

Fig. 65.

Let also $l$ = length of cord $AC$ or $BD$, $h$ = depth of $CD$ below $AB$, $W$ = mass of magnet in gm., so that $gW$ = weight of magnet in dynes, and let $T$ = tension in dynes of either cord. In the horizontal plane containing $CD$ let $A'B'$ be drawn equal and parallel and vertically under $AB$, and cutting $CD$ at its middle point $O$. Then the tension $T$ in $BD$ can be resolved into a vertical component $T.BB'/BD = Th/l$, and a horizontal component $T.DB'/l$ along $DB'$ ; and similarly for $T$ in $AC$.

Resolving vertically,

$$2Th/l = gW.$$

Taking moments about $O$, the couple exerted

$$= 2T.\{DB'/l\} \times \text{perpendicular from } O \text{ on } DB',$$
$$= 4T \times \text{area } BOD/l,$$
$$= 2Tab \sin \phi/l.$$

Substituting for $T$ its value found above, the couple

$$= gWab \sin \phi/h.$$

A couple can also be measured by the twist $\phi$ which it produces in a torsion wire (cf. § 4). Its value is $L\phi$, where $L$ is a constant for the wire.

**Ex. 1.**—A magnet 10·4 cm. long, ·6 cm. wide, and 45 gm. mass, oscillates in earth's horizontal field in a period of 12 seconds. Find its magnetic moment and pole strength.

$$\text{Period } T = 2\pi\sqrt{\frac{K}{MH}},$$

$$\therefore M = \frac{4\pi^2 K}{HT^2}.$$

Here

$$K = \tfrac{1}{12} \times 45\{10{\cdot}4^2 + {\cdot}6^2\} = 406{\cdot}8,\ T = 12,$$
$$\pi^2 = 9{\cdot}87,\ H = {\cdot}186,$$
$$\therefore M = 600.$$

The pole strength $m = M/10{\cdot}4$ approximately, but the divisor should be a little smaller than the true length. It is near enough to take $m = 600/10 = 60$.

**Ex. 2.**—In the above example, if the magnet be supported at a single point, find how far the centre of gravity will be from the vertical axis through that point. The earth's vertical field may be taken as ·48.

Let $h$ be the distance required. The moment of the weight about the point of suspension is $45gh = 45 \times 981 \times h$. The magnetic moment is nearly 600, hence the moment of the couple exerted in a vertical plane by $V$ is $600 \times {\cdot}44$.

For equilibrium,

$$45 \times 981 \times h = 600 \times {\cdot}44,$$
$$\therefore h = {\cdot}0060 \text{ cm.}$$

**Ex. 3.**—Show that if the magnet of Ex. 1, § 143, be suspended on a pivot so as to rest horizontal, this pivot must be at a distance about ·6 mm. from the centre of gravity, if the mass be 15 gm.

**Ex. 4.**—Show that a magnet can be suspended bifilarly so as to be in neutral equilibrium.

Taking the letters used above for bifilar suspension, let $AB$ be in the direction of the magnetic north, $N$ pointing southwards. For

any angular displacement the bifilar restoring couple is $gWab \sin \phi\, h$, and the magnetic couple is $MH \sin \phi$ tending to increase $\phi$. These are equal for all values of $\phi$ if

$$gWab = MHh.$$

A magnet suspended in this way, but with $gWab$ slightly greater than $MHh$, is in stable equilibrium when $\phi = o$, but is exceedingly sensitive to all disturbing magnetic fields.

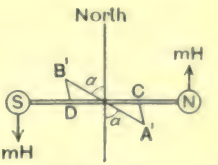


Fig. 66.

**Ex. 5.**—In a bifilar suspension $AB$ (Fig. 66) makes an angle $\alpha$ with the magnetic meridian while $CD$ is along the axis of the magnet as usual. Find the condition that the equilibrium position of the suspended magnet is perpendicular to the meridian.

Here the magnetic couple is $MH$, and the angle between $AB$ and $CD$ is $\frac{\pi}{2} - \alpha$; hence the condition of equilibrium is

$$MH = gWab \cos \alpha / h.$$

**Ex. 6.**—If the earth's field alters slightly by $\delta H$, find the angle $\theta$ through which above magnet turns.

Evidently

$$M(H + \delta H) \cos \theta = gWab \cos (\alpha - \theta)/h,$$
$$M(H + \delta H) = gWab\{\cos \alpha + \sin \alpha \tan \theta\}/h,$$
$$\therefore\ M\delta H = gWab \sin \alpha \tan \theta\, h,$$
$$\frac{\delta H}{H} = \tan \alpha \tan \theta.$$

An apparatus of this kind would register the alterations in the earth's field, if the magnet were provided with a tracing point adapted to record the changes of $\theta$ on a rotating cylinder.

**Ex. 7.**—With the above arrangement (Ex. 5), find the oscillation frequency.

For equilibrium

$$MH = gWab \cos \alpha\, h.$$

When $\alpha$ becomes $\alpha - \theta$, the restoring couple

$$= MH \cos \theta - gWab \cos (\alpha - \theta)\, h.$$

But $\theta$ is small, so neglecting $\theta^2$ we can put $\sin \theta = \theta$, $\cos \theta = 1$, and $\cos (\alpha - \theta) = \cos \alpha \cos \theta + \sin \alpha \sin \theta = \cos \alpha + \theta \sin \alpha$.

$$\therefore\ \text{Restoring couple} = -gWab\theta \sin \alpha\, h.$$

If $k$ be radius of gyration, moment of inertia

$$= Wk^2. \quad \therefore \text{ period} = 2\pi\sqrt{\frac{k^2h}{abg \sin a}}.$$

**148. Coulomb's Torsion Balance: Magnetic.**—A long magnet, with well defined poles at its ends, is supported horizontally in a stirrup depending from a vertical torsion wire. The arrangement is as in § 4, the magnet replacing the lever $BC$. There is a scale to read the position angle of the magnet, and the whole instrument should be adjusted so that the torsion head reads zero when the position angle is zero and the magnet is in the magnetic meridian. Of course no couple is acting, and the wire is therefore without twist.

A second magnet $ED$ can be set in a fixed position such that its pole $D$ would just touch the like pole $C$ if this were in its zero position defined as above.

Such contact will not occur, however, for $D$ will repel $C$ through a certain angle. This angle can be altered by turning the torsion head. Let its value be $\theta$ when the torsion head has been rotated through $a$ in the direction which would diminish $\theta$. Then the twist on the wire is $\theta + a$, and the couple due to it is $L(\theta + a)$.

The couple due to the earth's field is $MH \sin \theta$ (§ 145). The couple due to the repulsion acts the other way.

Let $l = OC = OD$, as in § 5. Then $CD = 2l \sin \frac{\theta}{2}$. If the pole strengths of $C$ and $D$ be $m_1$ and $m_2$, the repulsion is

$$m_1m_2/CD^2 = m_1m_2/4l^2 \sin^2 \frac{\theta}{2};$$

and its moment about $O$ is

$$m_1m_2 \cos \frac{\theta}{2} \Big/ 4l \sin^2 \frac{\theta}{2}.$$

Hence the equation of equilibrium is

$$L(\theta + a) + MH \sin \theta = \frac{m_1m_2 \cos \frac{\theta}{2}}{4l \sin^2 \frac{\theta}{2}}.$$

**149. Torsion Balance Experiments.**—If $\theta$ be small, the above formula reduces to

$$\alpha + A\theta = B/\theta^2$$

when $CD$ is inserted; and if $CD$ be not inserted we have

$$\alpha + A\theta = o.$$

First perform an experiment without $CD$. Observe $\theta$ (which should not exceed 10°) when a negative twist (of any magnitude) is given. For instance, when the torsion head is turned 90° let the magnet move 9°; we should infer $90 = 9A$, $A = 10$. Next insert $CD$ and find a series of corresponding values of $\alpha$ and $\theta$. Knowing $A$, we can tabulate $\alpha + A\theta$ against $\theta$, and verify $\alpha + A\theta \propto 1/\theta^2$, or can simply tabulate $\theta^2(\alpha + A\theta)$ against $\theta$, and show it is an approximate constant.

**Ex. 1.**—A torsion balance is in equilibrium position, $AB$ lying in the magnetic meridian with wire untwisted. The torsion head is turned 45°, and the magnet in consequence turns 10°. The magnet $CD$ is now inserted, and $AB$ is repelled to 25° from its zero position in which it would have touched $CD$. If the torsion head be turned back to zero, find the angle $\theta$ to which $AB$ points.

Use the formula

$$\alpha + A\theta = B/\theta^2.$$

We therefore have

$$-45 + 10A = o,$$
$$-45 + 25A = B/25^2,$$
$$A\theta = B/\theta^2.$$

We thus get

$$A = 4{\cdot}5,\ B = 42200 \text{ nearly},\ \theta^3 = B/A = 9380,\ \theta = 21.$$

**Ex. 2.**—Suppose that, as before, if no fixed magnet be inserted, the movable magnet moves 10° from the standard position if the torsion head rotates 45°. If the instrument be brought back to the standard position, and $CD$ inserted, let $AB$ be repelled 23°. Find how much the torsion head must be rotated to halve this angle.

We have

$$-45 + 10A = 0,$$
$$\therefore A = 4{\cdot}5, \text{ as before.}$$

Also when

$$\alpha = 0,\ \theta = 23,$$
$$\therefore 23A = B/23^2.$$

And $\alpha$ is required when

$$\theta = 11\tfrac{1}{2},$$

$$\therefore\ \alpha + 11\tfrac{1}{2}A = B/11\tfrac{1}{2}^2,$$

$$\therefore\ \alpha + 11\tfrac{1}{2}A = 4 \times 23A\ ;\ \alpha = 80\tfrac{1}{2}A = 362 \text{ nearly}.$$

**Ex. 3.**—A torsion balance is mounted on a turntable. Before taking any observation, the instrument is rotated as a whole, so as to bring the movable magnet $AB$ into the magnetic meridian. Under these conditions, if the fixed magnet repel the movable magnet 20° when the torsion head is at zero, how much must the torsion head be turned to halve the deflection?

Clearly the rotation of the instrument eliminates the earth's field. At the first experiment, the twist of the wire is 20°. At the second, the distance is halved, therefore the force is multiplied by 4, therefore the twist of the wire must be 80°. But 10° of twist is due to the position of the magnet, hence 70° of twist is put on by turning the torsion head.

**Ex. 4.**—If a turntable is used (as in Ex. 3) so that the magnet is always brought to rest in the magnetic meridian, show that $\alpha$ and $\theta$ satisfy the equation $\theta^2(\alpha + \theta) =$ constant.

**150. Field due to an Ideal Simple Magnet.**—Let $N$ and $S$ be the poles, $m$ and $-m$ their strengths. Let $C$ bisect $NS$, and $CN = CS = \frac{1}{2}l$. Let $M$ be the magnetic moment, so that $M = lm$. The field is required at any point $O$.

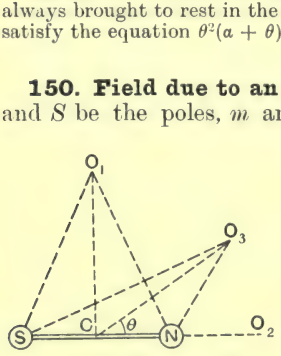

Fig. 67.

First let $O$ be at $O_1$, where $CO_1$ is perpendicular to $NS$. The magnet is then said to be placed "Broadside On" relative to $O$, or is in Gauss's B position. Let $f$ be the field due to a single pole $m$, at distance $r$, and let $r = NO_1 = O_1S$, and $d = CO_1$.

The poles $N$ and $S$ exert equal fields $f$ and $f$ at $O_1$ in directions $NO_1$, $O_1S$. These equal fields can be represented in magnitude and direction by the equal lines $NO_1$, $O_1S$; consequently the resultant $F_1$ is represented on the same scale by $NS$, and we have

$$F_1 = f.NS/O_1N = fl/r.$$

If we assume Coulomb's law,

$$f = m_1 m_2 / r^2,$$
$$\therefore F_1 = ml/r^3 = M/r^3.$$

This nearly equals $M/d^3$, if $l$ be short in comparison with $d$.

Next let $O$ be at $O_2$ in line $SN$; this is the "End On" position or Gauss's A position. Let $CO_2 = r$. Assume Coulomb's law. Then the field due to

$$N = m/NO_2^2 = m/(r - \tfrac{1}{2}l)^2 \ . \ . \ . \ .$$
$$S = m/SO_2^2 = -m/(r + \tfrac{1}{2}l)^2.$$

Total field is in direction $SN$, and its magnitude is

$$F_2 = \frac{m}{(r - \frac{1}{2}l)^2} - \frac{m}{(r + \frac{1}{2}l)^2}$$
$$= \frac{2mrl}{(r^2 - \frac{1}{4}l^2)^2} = \frac{2Mr}{(r^2 - \frac{1}{4}l^2)^2}.$$

If $l$ be small enough to be neglected,

$$F_2 = 2M/r^3.$$

Third, let $O$ be at $O_3$; $\angle NCO_3 = \theta$. The magnet (if short enough) can be resolved into one of moment $M \sin \theta$ perpendicular to $CO_3$, and $M \cos \theta$ along $CO_3$. If therefore $CO_3 = r$, we have two component fields,

$$M \sin \theta / r^3 \text{ perpendicular to } CO_3,$$

and

$$2 M \cos \theta / r^3 \text{ along } CO_3.$$

**Ex. 1.**—A magnet points eastward. Due east of it, at distance $r$ from its centre, is a compass needle. Find a formula for the angle $\phi$ through which the compass is deflected.

At the position $O_2$ of compass we have the earth's field $H$ northward, and the magnet's field $F = 2M/r^3$ eastward. Their resultant will therefore make such an angle $\phi$ with the north that $\tan \phi = F/H$. The compass indicates the direction of this resultant.

Hence

$$F = H \tan \phi = 2M/r^3.$$

**Ex. 2.**—A magnet points eastward. Due north of it, at distance $r$, is a compass. Find formula for angle of deflection $\phi$ of compass.

The compass is now at $O_1$ in Figure 67. It is affected by fields $H$ northward and $F$ westward.

As before,

$$F = H \tan \phi = M/r^3.$$

**Ex. 3.**—Use the above apparatus to prove Coulomb's law.

We showed that if $f$ be field due to one pole at distance $r$, then

$$F = lf/r \text{ in the B position.}$$

We want to show experimentally that

$$f \propto 1/r^2.$$

As in Ex. 2 above,

$$F = H \tan \phi,$$
$$\therefore lf/r = H \tan \phi.$$

Now by taking various distances and plotting $r^3$ against cot $\phi$ we verify that

$$\cot \phi \propto r^3, \text{ or } \tan \phi \propto 1/r^3,$$

and

$$\therefore f \propto 1/r^2,$$

which is Coulomb's law.

**Ex. 4.**—If the field due to a magnetic pole $\propto 1/r^n$, find the ratio of fields $F_2$ and $F_1$ of a magnet in "End On" and "Broadside On" positions.

$$F_2 = m/(r - \tfrac{1}{2}l)^n - m/(r + \tfrac{1}{2}l)^n,$$
$$= \frac{m}{r^n}\left\{\left(1 - \frac{l}{2r}\right)^{-n} - \left(1 + \frac{l}{2r}\right)^{-n}\right\},$$
$$= \frac{m}{r^n}\left\{\frac{nl}{r}\right\},$$

(using the binomial theorem, and rejecting higher powers of $l/r$)

$$= nM/r^{n+1},$$
$$F_1 = fl/r = lm/r^{n+1} = M/r^{n+1},$$
$$\therefore F_2 = nF_1 \text{ at the same distance.}$$

**Ex. 5.**—A short magnet points due east. Given magnetic moment $M$ and earth's horizontal field $H$ find the positions of the neutral points at which there is no resultant horizontal field.

Let $r$ be distance $CO$ and let $\theta$ be $\angle NOS$. $O$ is at $O_3$, Fig. 67. The field at $O$ has components

$M \sin \theta/r^3$ perpendicular to $CO$, making $\angle \theta$ with north,

$2M \cos \theta/r^3$ along $CO$, making $\angle 90° - \theta$ with north.

Resolving eastward

$$2M \cos^2 \theta/r^3 = M \sin^2 \theta/r^3,$$

$$\pm \tan \theta = \sqrt{2} = 1{\cdot}4142\,; \quad \theta = \pm 54°44'.$$

Resolving northward

$$H = -3M \sin \theta \cos \theta/r^3,$$

and

$$\sin \theta \cos \theta = \tan \theta/\sec^2 \theta = \tan \theta/(1 + \tan^2 \theta) = \pm \sqrt{2}/3,$$

hence

$$H = M\sqrt{2}/r^3.$$

**Ex. 6.**—To find the equation of the lines of force of a short magnet, neglecting earth's field.

Let the line of force through $O$ make $\angle \phi$ with $CO$. Take $C$ as origin and use polar coordinates; then

$$\tan \phi = r d\theta/dr.$$

But

$$\tan \phi = \text{field perpendicular to } r/\text{field along } r,$$

$$= \sin \theta/2 \cos \theta,$$

$$\therefore \; r d\theta/dr = \sin \theta/2 \cos \theta,$$

$$dr/r = 2 \cos \theta d\theta/\sin \theta.$$

Integrating,

$$\log r = 2 \log \sin \theta + \text{constant},$$

$$\therefore \; r = C \sin^2 \theta.$$

**Ex. 7.**—A compass needle is at $O$, due east of $C$. The magnet $NS$ is slowly rotated about $C$. Find the changes of slope of $O$.

When $NS$ is at slope $\theta$ to $CO$, the eastward field is $2M \cos \theta/r^3$ and northward field is $H - M \sin \theta/r^3$, therefore if $\phi$ be the slope of needle to the north

$$\tan \phi = 2M \cos \theta \{Hr^3 - M \sin \theta\},$$

$$= \frac{2 \cos \theta}{a - \sin \theta}\,; \; a \text{ being constant. If } a \text{ be large, } \tan \phi = 2 \cot \theta.$$

**Ex. 8.**—Find the field due to a short magnet $M$ at a point $P$ given by Cartesian coordinates.

Let $NS$ be magnet of length $l$, $C$ its centre. Let $PU$ be perpendicular on $SN$ (produced if necessary); and let

$$CU = x, \; UP = y.$$

The field due to $+m$ at $N$ is $m/PN^2$; its components are $m.NU/NP^3$ parallel to $CU$, $m.UP/NP^3$ along $UP$. Similarly $S$ gives components

$$-m.SU/SP^3, \; -mUP/SP^3.$$

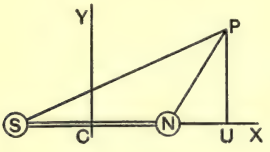


Fig. 68.

Hence if $X$ and $Y$ be fields along and perpendicular to magnet, and

$$r = CP = \sqrt{x^2 + y^2},$$

$$X = m. \left\{ \frac{NU}{NP^3} - \frac{SU}{SP^3} \right\},$$

$$= m \frac{x - \frac{1}{2}l}{\{(x - \frac{1}{2}l)^2 + y^2\}^{\frac{3}{2}}} - m \frac{x + \frac{1}{2}l}{\{(x + \frac{1}{2}l)^2 + y^2\}^{\frac{3}{2}}}.$$

Neglecting the square of $l$ and using binomial theorem,

$$X = \frac{mx}{r^3} \left\{ \left(1 - \frac{l}{2x}\right)\left(1 - \frac{lx}{r^2}\right)^{-\frac{3}{2}} - \left(1 + \frac{l}{2x}\right)\left(1 + \frac{lx}{r^2}\right)^{-\frac{3}{2}} \right\}$$

$$= \frac{mx}{r^3} \left\{ \frac{3lx}{r^2} - \frac{l}{x} \right\},$$

$$= \frac{M}{r^3} \left\{ \frac{3x^2}{r^2} - 1 \right\}.$$

In same way

$$Y = \frac{my}{r^3} \left\{ \left(1 - \frac{lx}{r^2}\right)^{-3/2} - \left(1 + \frac{lx}{r^2}\right)^{-3/2} \right\},$$

$$= \frac{3Mxy}{r^5}.$$

**Ex. 9.**—$SN$ is an ideal simple magnet, of finite length, whose pole strength is $m$. It points due eastward. Show that the conditions that $O$ shall be a neutral point are $ON^3/OS^3 = AN/AS$, and $H/m \,.\, OA = ON^{-3} - OS^{-3}$; $A$ being the foot of the perpendicular from $O$ on $SN$ produced.

**Ex. 10.**—$SN$ is a short magnet, centre $C$, and $O$ is a point due east of $C$. Find through what angle $\theta$ the magnet must be rotated from the meridian position in order to produce a given eastward field $X$ at $O$.

**Ex. 11.**—If $M$ be the magnetic moment of $SN$, arranged as in Ex. 10, and a compass needle be placed at $O$, find how the deflection $\phi$ of the compass needle varies as the magnet $SN$ is rotated. Distinguish the two cases when $M/CO^3 <$ and $> H$.

**Ex. 12.**—An ideal short magnet, pointed southward in the earth's field, has a neutral point at $O$. Find the field very near $O$, at a point whose coordinates $x$ and $y$, $N$ and $E$, are small compared to the distance of the magnet. Show the lines of force near $O$ are rectangular hyperbolae.

**Ex. 13.**—An ideal short magnet $SN$ is pointed northward in the earth's field. Show that there is a neutral point $O$ due east (or west) of the centre $C$ of the magnet. Find the field at places very near $O$ and show that the lines of force near $O$ are rectangular hyperbolae.

**Ex. 14.**—Due south of a compass needle is placed a magnet in the end-on position. The needle oscillates 20 times per minute. If the magnet be reversed, with the position of its centre unchanged, the needle oscillates about the same equilibrium position 4 times per minute. Find the period of oscillation if the magnet be removed, and the position of equilibrium if the magnet be rotated so as to point $E$ and $W$.

**Ex. 15.**—A short magnet oscillates in period 4 sec. if placed at a distance of 30 cm. due east of the centre of a strong magnet (B position). If placed at 20 cm. it oscillates in the reverse position in a period of 6 sec. Given $H = .18$, find the magnetic moment of the strong magnet.

**151. Equipotential Lines of Simple Magnet.** Exactly as in electrostatics, the potential due to a pole $m$ at distance $r = m/r$.

The potential at $P$ due to a magnet $NS$ whose poles are $+ m$ and $- m$ is therefore

$$\frac{m}{NP} - \frac{m}{NS}$$

Hence the equipotential lines are the loci for which

$$1/NP - 1/SP = \text{constant}.$$

We can draw these equipotentials by geometrical construction. For instance, let $SN = 10$ cm., and $m = 10$ units. Required to construct the equipotential for potential $= 2$.

$$10/NP - 10/SP = 2.$$

The work is identical with that in Ex. 3, § 13.

For an indefinitely short magnet, using previous notation, the potential $V$ at $P$

$$= m\{1/NP - 1/SP\} = m\{SP - NP\}/SP \,.\, NP.$$

But if

$$CP = r \text{ and } NCP = \theta,$$
$$SP - NP = l \cos\theta \text{ nearly,}$$

and

$$SP \,.\, NP = r^2,$$
$$\therefore V = ml \cos\theta/r^2 = M \cos\theta/r^2.$$

**152. Lines of Force of Simple Magnet.**—The whole theory developed for Electrostatics in §§ 14-16 applies directly to Magnetism.

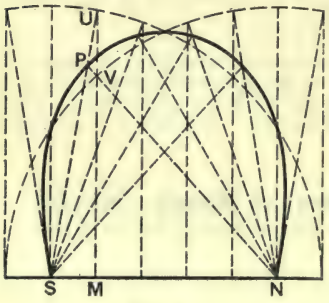

Fig. 69.

The form of lines of force for a simple magnet is found as follows:—

With letters as in Fig. 69 the fields along $PN$, $SP$ are

$$m/r^2,\ m/r'^2.$$

If $PT$ be tangent, resolving perpendicular to $PT$ we have

$$\phi,\ \phi' = NPT,\ SPT, \text{ if } m \sin\phi/r^2 = m \sin\phi'/r'^2.$$

Now by differential calculus, if $ds$ be element of arc,

$$\sin\phi = rd\theta/ds,$$
$$\sin\phi' = -r'd\theta'/ds,$$
$$\therefore\ d\theta/r + d\theta'/r' = 0.$$

But

$$r : r' = \sin\theta' : \sin\theta,$$
$$\therefore \sin\theta\, d\theta + \sin\theta'\, d\theta' = 0,$$
$$\therefore \cos\theta + \cos\theta' = \text{constant.}$$

For any one line of force let this constant $= SN/R$. Draw about $S$ and $N$ two circles of same radius $R$ and draw

a set of parallel ordinates $MUV$ perpendicular to $SN$ cutting the circles in $U$ and $V$. For each position of $M$, $SU$ and $NV$ determine a point $P$ on the curve.

For obviously

$$\cos\theta' = SM/R, \ \cos\theta = MN/R,$$
$$\therefore \cos\theta + \cos\theta' = SN/R = \text{constant}.$$

This construction is given in Minchin and Dale, *Mathematical Drawing.*

**Example.**—A magnet points northward. Two lines of force are traced from the $N$ pole, one each side of the north line. At one distance they are 1 mm. apart, at another 3 mm. Compare the fields at these places.

The sections of the same tube of force at these two distances are circles of diameter 1 and 3 cm. ; so that the areas are as 1 to 9. Hence (§ 22) the fields are as 9 to 1.

**153. Doublets.**—A short simple magnet is called a Doublet, in the limit when its length is quite negligible.

Since $M = ml$, $m$ must be very large if $l$ is negligibly small and $M$ finite.

The theorems stated as approximations for ideal simple magnets can be taken as absolutely true for doublets, namely,

The Potential $= M\cos\theta/r^2$,

The Field at distance $r$

$$= 2M/r^3 \text{ if } \theta = 0,$$
$$= M/r^3 \text{ if } \theta = 90^\circ.$$

and in general has components

$$2M\cos\theta/r^3 \text{ radial},$$

and

$$M\sin\theta/r^3 \text{ transverse}.$$

Or

$$\frac{M}{r^3}\left(3\frac{x^2}{r^2} - 1\right) \text{ parallel to axis of doublet},$$

and

$$\frac{3Mxy}{r^5} \text{ perpendicular to it}.$$

An ideal simple magnet, whose length $l = SN$ and pole strength $= m$, can evidently be replaced by a set of shorter

equal magnets of pole-strength $m$ and lengths which collectively make up $SN$. If $S$ and $N$ be joined by a curved line, this can be divided into equal short arcs $s$ and each regarded as a simple magnet of length $s$, pole-strength $m$, and moment $sm$. Thus a simple magnet is equivalent to a string of doublets arranged longitudinally.

A compound magnet can always be regarded as the resultant of a set of simple ones. It is therefore resolvable into strings of doublets.

Doublets can be utilised with advantage in certain problems. (See §§ 219-221.) We shall also find that they have real physical significance. The molecules of iron and other magnetic substances are doublets. Hence a magnet is actually a group of an enormous number of doublets.

**154. Oscillation Magnetometers.**—Magnetometers are instruments used to measure either magnetic fields or magnetic moments or pole strengths of magnets.

If we time the oscillations in a field of strength $H$ of a magnet whose magnetic moment is $M$ and moment of inertia $K$, then the period $T$ is given by

$$T = 2\pi \sqrt{K/MH} \text{ (§ 146).}$$

This formula can be used to measure either $M$ if $H$ is given or $H$ if $M$ is given. It primarily measures their product, for

$$MH = 4\pi^2 K/T^2.$$

**155. Deflection Magnetometers (Tangent).**—A small subsidiary magnet $ns$ (Fig. 70) called a compass needle is acted on by the earth's horizontal field $H$. It is disturbed by the horizontal field $F$ of the magnet experimented on. Its direction then shows the resultant of $F$ and $H$.

If this magnet be placed with its centre $C$ due east of the compass needle, and be directed towards it, the compass is acted on by fields $F$ and $H$ westward and

northward, and will point in the direction of their resultant. This makes angle $\theta$ with the north, where

$$\tan \theta = F/H.$$

But

$$F = 2M/r^3, \text{ if } r = CO \text{ in figure.}$$
$$\therefore M/H = \tfrac{1}{2} r^3 \tan \theta.$$

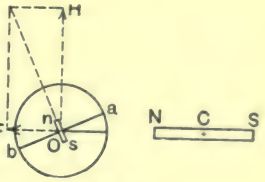

Fig. 70.

In figure, *ns* is shown carrying a light index *ab* perpendicular to its length, whereby the angle $\theta$ is more easily read.

If, however, the magnet is placed due south of the compass needle (Fig. 71) and directed eastward, its field $F$ is again due west, and

$$\tan \theta = F/H.$$

But

$$F = M r^3,$$
$$\therefore M/H = r^3 \tan \theta.$$

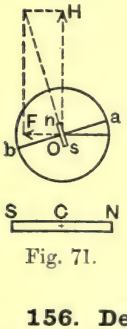

Fig. 71.

Both these arrangements can be called Tangent Magnetometers. Instead of the earth's horizontal field we can have an artificial field either stronger or weaker, to vary the sensitiveness of the instrument. We must understand North to mean the direction of this field unmodified by the magnet *SN*.

**156. Deflection Magnetometers (Sine.**—If the magnet $M$ is pointed towards the compass needle (Gauss position A) in an arrangement which can be rotated about an axis, and we rotate until the compass points perpendicular to the line joining it to $M$, the resultant of $F$ and $H$ will be perpendicular to $F$ and therefore, if $\theta$ be slope of compass to the north line,

$$\sin \theta = F/H,$$

and

$$F = 2M r^3,$$
$$\therefore M/H = \tfrac{1}{2} r^3 \sin \theta.$$

If $M$ be placed in a similar arrangement, but in the B position, we should have

$$M/H = r^3 \sin\theta.$$

These arrangements can be called Sine Magnetometers.

**157. Precautions and Corrections.**—In using a sine or tangent magnetometer we must remember the magnet used is not an ideal simple one, and its length is not always negligible.

If $l$ be its length, and it be used in the end on position, the correct formula is

$$F = 2Mr/(r^2 - \tfrac{1}{4}l^2)^2 = \frac{2M}{r^3}\left\{1 - \frac{l^2}{4r^2}\right\}^{-2},$$

but in a tangent magnetometer

$$\tan\theta = F/H,$$

$$\therefore \frac{M}{H} = \frac{r^3}{\tan\theta}\left\{1 - \frac{l^2}{4r^2}\right\}^2.$$

If we expand by binomial theorem and neglect $l^4$,

$$\frac{M}{H} = \frac{r^3}{\tan\theta}\left\{1 - \frac{l^2}{2r^2}\right\};$$

or

$$\frac{H}{M} = \frac{\tan\theta}{r^3}\left(1 + \frac{l^2}{2r^2}\right).$$

Again, the geometrical centre of the magnet is not necessarily midway between its $N$ and $S$ poles. In using a tangent magnetometer end on, one corrects for this by observing the deflection $\theta$ when $SN$, pointing eastward, is placed in positions $A$ and $B$ due east and west (magnetic) of compass, and when $SN$ is at $A$ and $B$ pointing westward; the geometrical centre being at same distance $r$ in all four cases. The true centre is of course too near in two positions and too far in the other two. We get four values of $\theta$; or we may read eight values, for $\theta$ is marked on a graduated scale by a long pointer $ab$ both ends of which can be read. The mean of all these readings is practically reliable.

**Ex. 1.**—Find the correction depending on the length of the magnet when it is used in the B position.

The exact formula is

$$F = M/(r^2 + \tfrac{1}{4}l^2)^{3/2},$$

and if we neglect $\left(\frac{l}{r}\right)^4$ this gives

$$\frac{F}{M} = \frac{1}{r^3}\left\{1 - \frac{3}{8}\cdot\frac{l^2}{r^2}\right\}; \quad \frac{M}{F} = r^3\left(1 + \frac{3}{8}\cdot\frac{l^2}{r^2}\right).$$

**Ex. 2.**—A magnet of length 10·4 cm., width 1 cm., and rectangular section, has mass 55 gm. If suspended horizontally it oscillates about the meridian with period 14·2 sec. If placed broadside on in a sine magnetometer at distance 30 cm. it produces a deflection (averaged) of 6°. Find the earth's horizontal field $H$, and the magnetic moment of the magnet.

From the length and mass, we deduce that the moment of inertia

$$K = 55 \times (10{\cdot}4)^2/12 = 495{\cdot}7.$$

If $T$ be the period, by § 146

$$T = 2\pi\sqrt{K/MH},$$
$$\therefore\ MH = 4\pi^2 K/T^2 = 4\pi^2 \times 495{\cdot}7/(14{\cdot}2)^2$$
$$= 97{\cdot}05.$$

In the sine magnetometer

$$M/r^3 = H\sin\theta,$$
$$M/H = r^3\sin\theta = 27000 \sin 6° = 2822,$$
$$\therefore\ H^2 = 97{\cdot}05/2822 = {\cdot}03439,$$

and

$$H = {\cdot}185 \text{ unit of field.}$$

Hence also

$$M^2 = 97{\cdot}05 \times 2822 = 273900,$$
$$M = 520 \text{ units of magnetic moment.}$$

A somewhat more accurate value might be obtained by introducing the corrections for the length of the magnet. (Ex. 1.)

**Ex. 3.**—In a Kew magnetometer the magnet oscillates with period 6·6 seconds. If a brass bar of magnetic moment 500 be attached to it, they oscillate with period 7·5. The magnet placed broadside on at 40 cm. distance produces average deflection 4·5°, the arrangement being that of a sine magnetometer. Calculate $H$. Calculate also the magnetic moment of the magnet and its moment of inertia.

Let $K$ be the moment of inertia of the magnet, then

$$T = 2\pi \sqrt{K/MH}, \quad MH = 4\pi^2 K/T^2,$$
$$\therefore \ MH = 4\pi^2 K/6{\cdot}6^2, \ 4\pi^2 K = 6{\cdot}6^2 MH.$$

For the magnet and bar,

$$4\pi^2(K + 500) = 7{\cdot}5^2 MH.$$

Hence

$$(K + 500)/K = 7{\cdot}5^2/6{\cdot}6^2, \text{ giving } K = 1716,$$

and

$$4\pi^2 . 500 = (7{\cdot}5^2 - 6{\cdot}6^2)MH = (7{\cdot}5 + 6{\cdot}6)\,(7{\cdot}5 - 6{\cdot}6)MH,$$
$$= 14{\cdot}1 \times {\cdot}9\,MH, \text{ giving } MH = 1555.$$

As in previous example

$$M/H = r^3 \sin\theta,$$
$$= 40^3 \sin 45^\circ = 45260.$$

Hence

$$H^2 = 1555/45260, \qquad H = {\cdot}185,$$
$$M^2 = 1555 \times 45260, \quad M = 8390.$$

**Ex. 4.**—In Paris, $H = {\cdot}20$. A magnet has magnetic moment 1240 and moment of inertia 2600. Calculate its period of horizontal oscillation, and the deflection it would produce in a compass magnetometer if placed at 20 cm. in the usual end-on and broadside-on positions, the magnetometer not being rotated.

$$T = 2\pi \sqrt{K/MH} = 2\pi \sqrt{2600/1240 \times {\cdot}20} = 20{\cdot}34 \text{ sec.}$$

In the end-on position,

$$2M/r^3 = H \tan\theta,$$
$$\therefore \tan\theta = 2M/r^3 H = 2 \times 1240/(20)^3 \times {\cdot}2\,; \ \theta = 57^\circ\ 10'.$$

In the broadside-on position,

$$M/r^3 = H \tan\theta,$$
$$\therefore \tan\theta = M/r^3 H = 1240/20^3 \times {\cdot}2\,; \ \theta = 37^\circ\ 3'.$$

**Ex. 5.**—Two magnets fastened together, so that both point horizontally and parallel and are centrally suspended, oscillate in period 14 sec. If one of them be fastened to the other with its position exactly reversed, the period is 36 sec. If they be fixed (horizontally and centrally) with their axes at right angles, what will the period be?

In all three cases the moment of inertia of the combination has the same value $K$, hence the square of the period $T$ is inversely proportional to the magnetic moment $M$; or $T^2 = C/M$, where $C$ is a constant. If $A$ and $B$ be the magnetic moments of the separate magnets, $M$ has the values

$$A + B,\ A - B,\ \sqrt{A^2 + B^2}$$

in the three cases. But

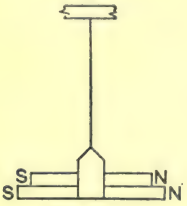

Fig. 72.

$$M = C/T^2,$$
$$\therefore\ A + B = C/14^2,$$
$$A - B = C/36^2.$$

Square and add,

$$2(A^2 + B^2) = C^2\{1/14^4 + 1/36^4\}.$$

But the unknown period $T$ is given by

$$T^2 = C/\sqrt{A^2 + B^2},$$
$$\therefore\ T^4 = 2/\{1/14^4 + 1/36^4\},$$
$$T = 16{\cdot}56 \text{ sec.}$$

**Ex. 6.**—Show that if a certain condition be satisfied it is possible to fix three horizontal magnets together so that the system is quite unaffected by the earth's field. How can this be done in practice?

The system must have no resultant magnetic moment. This is the case if the magnetic moments of the magnet be represented in magnitude and direction by the three sides of a triangle taken in order (by § 144). The magnets can be fixed in this way provided no one has magnetic moment greater than the sum of the other two. In practice one would adjust by turning each magnet successively in such a way as to increase the oscillation period. When this was practically infinite, it always $= 2\pi\sqrt{I/MH}$, so $M$ would necessarily be zero.

**Ex. 7.**—A magnet is suspended in a wooden case. Its motions are shown by means of a mirror attached to it, which throws a spot of light perpendicularly to the centre of a horizontal scale at a distance of 50 cm. Find the eastward or westward magnetic field necessary to move the spot of light 1 cm. [$H = {\cdot}186$.]

If the magnet turns through angle $\theta$, the reflected ray turns $2\theta$.

$$\therefore\ 50 \tan 2\theta = 1,$$
$$\tan 2\theta = 1/50.$$

Hence $$\tan\theta = 1/100 \text{ nearly};$$

but $$\tan\theta = F/H,$$
$$\therefore\ F = {\cdot}00186.$$

**Ex. 8.**—The above magnet has oscillation period 3·5 sec. A control magnet is used, which brings the spot to centre of scale as before but alters the period to 7 sec. Find now the magnetic field needed to move the spot 1 cm.

The period is doubled, hence the magnetic field is multiplied by 1/4 and = ·186 ÷ 4 = ·0465. The field necessary to produce the deflection = ·000465.

**Ex. 9.**—A magnet is suspended on a horizontal axis perpendicular to its length whose height above the centre of gravity is $y$. It rests horizontal. If $w$ be its weight, $M$ its magnetic moment, find the horizontal distance between axis of suspension and centre of gravity. Take $V$ for vertical field. Find also through what angle the magnet will tilt if $V$ alters by $\delta V$ (small).

If $x$ be horizontal distance,

$$xw = MV.$$

If $V$ alters to $V + \delta V$, the extra couple is $M\delta V$. The magnet swings through a small angle $\theta$, therefore its centre of gravity moves $y\theta$ horizontally, and the moment of the weight alters by $wy\theta$. Hence

$$wy\theta = M\delta V;$$
$$\theta = M\delta V/wy.$$

This arrangement is used in a recording magnetometer to measure changes in the earth's vertical field.

**Ex. 10.**—A magnet is suspended so as to rest vertical, the weight and magnetic moment being $w$ and $M$. Show that, if the axis of suspension point due east and west and be at a horizontal distance $x$ and vertical distance $y$ from the centre of gravity,

$$xw = HM,$$

and that, if $H$ alters by $\delta H$, the magnet turns through an angle (small) $M\delta H/wy$.

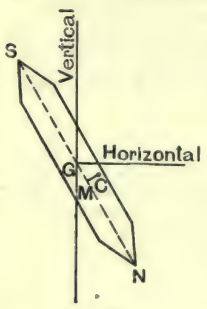

Fig. 73.

**158. Dip Needle.**—If a magnet be suspended on a horizontal axis which points accurately east and west, and the axis contains the centre of gravity of the magnet, the magnet will point in the direction of the earth's resultant field, and its slope = the dip $\delta$ (§ 141). If, however, the coordinates of the axis of suspension $C$, relative to the centre of gravity $G$, be $\alpha$ in direction of magnetisation and $\beta$ perpendicular to this direction, let us find the difference $\theta$ produced by the weight $w$ of the magnet between the actual slope and the dip.

If $GM$, $MC$ be $\alpha$ and $\beta$, and if $\delta - \theta$ be the slope of $GM$ to the vertical, the horizontal distance from $C$ to $G$ is

$$\alpha \cos(\delta - \theta) + \beta \sin(\delta - \theta).$$

Hence the moment of the weight is

$$\{\alpha \cos(\delta - \theta) + \beta \sin(\delta - \theta)\}w.$$

If $R$ be the earth's resultant field, it makes $\angle\theta$ with the axis of the magnet, and exerts a couple $RM \sin\theta$.

Hence for equilibrium,

$$RM \sin\theta = \{\alpha \cos(\delta - \theta) + \beta \sin(\delta - \theta)\}w.$$

If we regard $\alpha$, $\beta$, $\theta$ as very small, we have

$$RM\theta = \alpha w \cos\delta' + \beta w \sin\delta',$$

where

$$\delta' = \delta - \theta,$$

and is the slope of the dip needle.

**159. Deviation.**—This is the angle between the magnetic north (direction of $H$) and the true north. It is determined by the magnetic compass.

This consists of a light circular lamina called the Card, turning centrally on a vertical needle. To its lower part are fastened a number of separate small magnets; this gives much greater magnetic moment for the same weight than a single larger one. It is balanced to hang horizontally under the influence of its weight and the earth's vertical field $V$; if $V$ alters (as when the compass travels to another country) the balance is corrected by small counterpoises. Instead of this form of suspension the lamina can be floated on a liquid, and this ensures perfect frictionless adjustment. The upper part of the card is graduated in degrees, and sometimes in "points" or eighth parts of a right angle from the $N$ point on card. The containing case carries an index, which may be a lens, to read the position of the card by. When the case is in standard position, the reading is 0, or no "points," from the magnetic north.

Sometimes (as in the prismatic compass) a telescope or sights are attached to the case, which is then free to rotate. The compass reads the angle between that vertical plane in which the telescope or sights are pointed and the north (magnetic). It thus gives the magnetic "bearing" or azimuth of the object viewed. Sometimes (as in a ship's compass) the compass case is fixed. The reading is then to determine the bearing or azimuth of a line parallel to its zero radius.

# CHAPTER IX.

## MAGNETIC FIELD OF CURRENTS.

**160. Ampere's Law.**—The law is stated as follows. The magnetic field exerted at a point by a short segment of a conductor carrying a current is directed in a line perpendicular to the plane containing the point and the segment. Its magnitude is proportional jointly to the current, to the length of the segment resolved perpendicular to the line joining it to the point, and to the inverse square of this joining line.

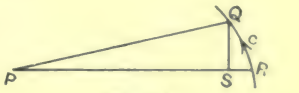

Fig. 74.

Such experiments as have hitherto been made are in agreement with consequences mathematically deduced from this law. But these experiments are on complete electric circuits. It is practically impossible to isolate experimentally a portion of a circuit, so as to study its magnetic field.

To deduce a formula. Let $P$ be the point, $QR$ the segment, and $QS$ a perpendicular from $Q$ on $PR$. Let $C$ be the current, in any arbitrary units. In the limit, when $QR$ is indefinitely small, the magnitude of the field is

$$= A.c.QS/PQ^2, \text{ where } A \text{ is a constant.}$$

If $\theta$ be the circular measure of angle $QPS$, $\theta = QS/PQ$.

$$\therefore \text{ the field } = Ac\theta/PQ.$$

The direction of the field is perpendicular to the plane $PQR$, and so that it is towards the observer if the motion from $Q$ to $R$ is seen to be counterclockwise around $P$. If therefore the angle $\theta$ be positively described, the field is upward through the figure (*i.e.* towards the observer).

**161. Absolute Unit of Current.**—The unit is so chosen as to simplify the above formula. It is defined as such a current that if it flowed in a circle of 1 cm. radius it would exert a magnetic field $2\pi$ at the centre of the circle.

Hence $F = 2\pi$ when $r = 1$ and $c = 1$. But $F = 2\pi Ac/r$, consequently $A = 1$. If therefore we express current in terms of this unit, Ampere's law is given by the formula

$$F = c\theta/r.$$

It may be taken as a numerical fact that, very approximately, the unit current = 10 commercial amperes (§ 88).

**Ex. 1.**—Find the magnetic field at centre $O$ (Fig. 75) of a circular coil of $n$ turns, radius $r$, when the current flowing is $c$ absolute units.

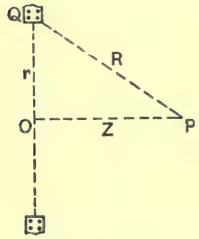

Fig. 75.

The field due to one turn is $2\pi c/r$, hence that due to $n$ turns is $2\pi nc/r$.

Some authorities define the **Coil Constant** as the field produced at the centre per unit current. Its value is evidently $2\pi n/r$.

**Ex. 2.**—If the above coil be set in the magnetic meridian, find through what angle it would deflect a compass needle situated at the centre of the circle.

The field due to coil is

$$F = 2\pi nc/r,$$

directed perpendicular to the coil and therefore eastward or westward. The earth's horizontal field is $H$, northward. The resultant of $F$ and $H$, at right angles to one another, makes an angle $\theta$ with $H$ such that

$$\tan\theta = F/H.$$

Hence the deflection is given by

$$\tan\theta = 2\pi nc/rH.$$

Since $c = \dfrac{rH}{2\pi n}\tan\theta$, this arrangement can be used to measure currents. It is the usual type of tangent galvanometer. The coefficient $c_0 = \dfrac{rH}{2\pi n}$ is the **Galvanometer Constant.** It must not be confounded with the **Coil Constant** $\dfrac{2\pi n}{r}$.

If $C$ be the current in amperes,

$$C = 10c,$$

and can be put

$$= C'_0 \tan \theta$$

where

$$C'_0 = 5rH/\pi n.$$

**Ex. 3.**—Find the galvanometer constant of a tangent galvanometer of 10 turns, of radius 12 cm.

$$c_0 = rH/2\pi n = 12 \times \cdot 186/20\pi = \cdot 0355 ; \ C'_0 = \cdot 355.$$

The constant is ·355 ampere.

**Ex. 4.**—Find the constant of an instrument of 500 turns, mean radius 3 cm.

$$Ans. \ C_0 = \cdot 0018 \text{ in amps.}$$

**Ex. 5.**—Find field exerted by current $c$ in a circular coil of $n$ turns and radius $r$ at a point on the axis of the coil at a distance $z$ from its centre.

Let $P$ be the point (Fig. 75), $O$ centre of coil ; $OP = z$.

Let $Q$ be point on circumference, $OQ = r$, $PQ = R$.

The field at $O$ due to short arc $s$ is $sc/R^2$ perpendicular to $PQ$ and the arc. Its resolute $F$ in line $OP$

$$= sc \sin QPO/R^2 = scr/R^3.$$

The resolutes perpendicular to $OP$ cancel one another, by symmetry. The sum of resolutes along $OP$ is deduced by noticing that the sum of all the small arcs $s$ is $2\pi rn$,

$$\therefore \ F = 2\pi r^2 nc/R^3 = 2\pi r^2 nc/(r^2 + z^2)^{3/2}.$$

[At great distances $F = 2\pi nr^2 c/z^3$, the coil behaves like an end-on magnet of magnetic moment $\pi nr^2 c$. See § 150.]

**Ex. 6.**—Find the constant of a tangent galvanometer in which the compass needle is supported on axis of circular coil, but not at its centre.

As above, in Exs. 2 and 5,

$$F = H \tan \theta = 2\pi r^2 nc/R^3,$$

$$\therefore \ c = R^3 H \tan \theta/2\pi r^2 n,$$

$$\therefore \ c_0 = H(r^2 + z^2)^{3/2}/2\pi r^2 n.$$

**Ex. 7.**—Find the value of $z$ in order that the sensitiveness may be 1/10 as great as at the centre of the coil.

We get

$$(r^2 + z^2)^{3/2} = 10r^3 ; \ z^2 = r^2(10^{2/3} - 1),$$

whence finally

$$z = 1{\cdot}908r.$$

**Ex. 8.**—Find the constant of a Helmholtz galvanometer. In this, $\frac{1}{2}n$ turns are in one coil of radius $r$, and $\frac{1}{2}n$ in another identical coil; the coils are placed coaxial and parallel at distance $r$ apart, and the compass is at $A$ (Fig. 76) halfway between their centres on their common axis.

Fig. 76.

Evidently the field is the same as with one coil of $n$ turns, if $z = \frac{1}{2}r$. The constant $= 5\sqrt{5}rH/16\pi n$.

**Ex. 9.**—A conducting ring of radius $a$ is charged with $Q$ electrostatic units and rotated in its own plane about its centre with angular velocity $w$. Find the magnetic field at a point in its axis of rotation at distance $z$ from its centre.

$Q$ electrostatic units is $Q/v$ electromagnetic units (§ 82). The velocity is $aw$, hence the current is

$$c = Qaw/v,$$

and the magnetic field

$$= 2\pi ca^2/(a^2 + z^2)^{3/2} = 2\pi a^3 wQ/v(a^2 + z^2)^{3/2}.$$

It can easily be shown that the electrostatic field is in the same line as the electromagnetic, and

$$= Qz/(a^2 + z^2)^{3/2}.$$

Of course the electromagnetic field would be exerted on unit north pole, and affects only magnetic matter. The electrostatic field affects only electric charges.

**Ex. 10.**—Find the force exerted by the coil of Ex. 5 on a small magnet of moment $M$, placed longitudinally on its axis at a distance $z$ from its centre.

Let $m$ be pole strength and $l$ length, so that $lm = M$. In the limit we shall make $l = o$. We can consider the north pole $+m$ to be at a distance $z$, and the south pole at distance $z + l$. Hence, using formula of Ex. 5, the total force

$$= 2\pi r^2 nc\left[\frac{m}{(r^2 + z^2)^{3/2}} - \frac{m}{\{r^2 + (z + l)^2\}^{3/2}}\right],$$

$$= \frac{2\pi r^2 ncm}{(r^2 + z^2)^{3/2}}\left[1 - \left(1 + \frac{2lz + z^2}{r^2 + z^2}\right)^{-\frac{3}{2}}\right],$$

$$= \frac{2\pi r^2 ncm}{(r^2 + z^2)^{3/2}} \cdot \frac{3lz}{r^2 + z^2}, \text{ neglecting } l^2,$$

$$= \frac{6\pi r^2 ncMz}{(r^2 + z^2)^{5/2}}.$$

**Ex. 11.**—Copper wire of 1 mm. radius is made into a ring of 10 cm. radius. It is charged with 100 electrostatic units, and rotated with a linear velocity of 100 metres per second. Find the current in amperes, and the field at the centre of the coil. Assume that the charge is carried with the wire.

A charge 100 moving with velocity 100,000 is a current of $100 \times 100{,}000 = 10^7$ E.S. units. This is $10^7/3 \times 10^{-10} = 1{\cdot}3000$ E.M. units. The radius is 10, therefore the field at centre $= 2\pi \cdot 3000 \div 10 = {\cdot}00021$ unit of field. The current is 1·300 amperes.

**Ex. 12.**—Find the galvanometer constant of a tangent galvanometer of 300 turns, the diameter of the coil being 16 cm.

**Ex. 13.**—A small galvanometer has a coil of 50 turns of mean radius 2 cm. Its needle carries a mirror, and the deflection is read by a beam of light reflected to a scale at a distance of 50 cm. Find in amperes the current indicated per cm. length on the scale.

**Ex. 14.**—A tangent galvanometer to measure moderate currents with great accuracy has a single turn of diameter 20 cm. Find the galvanometer constant (1) when the indicating compass needle is at the centre, (2) when it is 24 cm. from the centre, on the axis of the coil.

**Ex. 15.**—Find the constant of a Helmholtz tangent galvanometer of 20 turns, radius 15 cm.

**Ex. 16.**—A galvanometer has two distinct windings, of radii 9 and 7. The inner has 20 turns. When a current is passed in series through both, the deflection is 25°. When the inner is reversed, the deflection is 11° the same way. Find the number of turns of the outer winding.

**Ex. 17.**—A galvanometer has 500 turns of radius 10; and its resistance is 220 ohms. Find the voltage indicated between its terminals by a deflection of 45°.

**Ex. 18.**—A galvanometer is in series with a voltameter. Its deflection is kept at 45°, and 1 c.cm. of hydrogen is given off per minute at normal temperature and pressure. Find the constant of the galvanometer. If its radius be 6 cm., find the number of turns.

**162. Field of a Rectilinear Current.** Let $P$ be the point at which field is required, and $QR$ be an element of current in the infinite straight line $XY$. Drop a perpendicular $PA$ on the straight line, and let a circle of

centre $P$ and radius $PA$ cut $PQ$ and $PR$ in $q$ and $r$. Draw the diameter $xPy$ parallel to $XY$, drop perpendiculars $rn$ and $qm$ on it, and draw $rs$ perpendicular to $qm$, then ultimately the triangles $qsr$, $QAP$ are similar.

The field at $P$ due to $QR$

$$= \frac{c.\angle RPQ}{PQ} = \frac{c.qr}{PQ.PA},$$

$$= \frac{c.rs}{PA^2}, \text{ for } qr : rs = PQ : PA,$$

$$= c.mn/PA^2.$$

The sum of all the elements $mn$ corresponding to all the

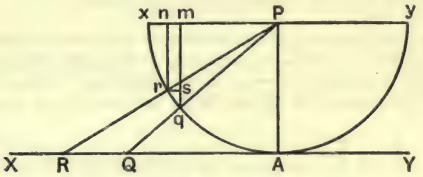


Fig. 77.

elements $PQ$ of the infinite straight line is obviously $= xy = 2.PA$. Hence the field due to an infinite straight line

$$= 2c/PA.$$

It is therefore proportional to the inverse distance.

**Field of a Finite Straight Line.**—The sum of portions $mn$ corresponding to elements of a finite straight line $AR$ is obviously equal to $Pn$

$$= PA \sin APR.$$

Hence the field due to $AR$

$$= c.\sin APR/PA.$$

Any portion of any straight line can be represented as the sum or difference of two portions like $AR$.

**Ex. 1.**—A current $c$ flows round a rectangle $PQRS$ whose sides $PQ$ and $RS$ are $a$ and $b$. Find the field at centre of rectangle.

With letters as in figure, the field due to $PA$

$$= \frac{c \sin POA}{OA} = \frac{ca}{\sqrt{a^2 + b^2}} \cdot \frac{2}{b}.$$

Similarly for $PB$, and like segments.

Hence the field of rectangle

$$= \frac{8c}{\sqrt{a^2 + b^2}} \left( \frac{a}{b} + \frac{b}{a} \right),$$

$$= \frac{8c\sqrt{a^2 + b^2}}{ab}.$$

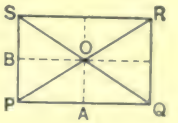

Fig. 78.

$\left[ \text{When } a \text{ is infinite, this } = \frac{8c}{b}. \right]$

**Ex. 2.**—If the length $AR = e$, and $PA = z$ (Fig. 77), show that the field at $P$ due to $c$ in $AR$ is equal to $ce/z\sqrt{z^2 + e^2}$.

**Ex. 3.**—Find the field due to a circuit in the form of an equilateral triangle, at the centre of the triangle.

**Ex. 4.**—Find the field at centre due to a regular polygon of $n$ sides. Show that this agrees with the formula for a circular current in the limit when $n$ is infinite. [Field $= cp/rR$ where $p$ is perimeter and $r$, $R$ are radii of inscribed and circumscribed circles.]

**Ex. 5.**—Show that the field at a distance $x$ from the centre of a regular polygon, in a direction perpendicular to its plane, is

$$cpr/(r^2 + x^2)\sqrt{R^2 + x^2};$$

the notation being as in Ex. 4.

**163. Potential due to an Infinite Linear Current.**—Let it be required to find an expression for the potential at a point $P$ due to a current $c$ in an infinite straight line. The infinite straight line which carries the current is supposed to be perpendicular to the plane of the figure and to cut this plane at $A$. The field at any point

$Q$ near $P$ is $2\ c/AQ$ perpendicular to $AQ$, and its component in the direction $QP$ is $2c \sin AQP/AQ$. The work done on unit $N$ pole travelling from $Q$ to $P$

$$= 2c.QP \sin AQP/AD,$$

$$= 2c \sin QAP, \text{ by Trigonometry,}$$

$$= 2c \times \text{circular measure of angle } QAP, \text{ in limit.}$$

If an arbitrary point $X$ is chosen for zero of potential, let any arc $XQP$ be described. For each small portion such as $QP$, the work done on unit pole $= 2c \times$ angle subtended at $A$. Hence for the whole path $XP$ the work

$$= 2c \angle PAX,$$

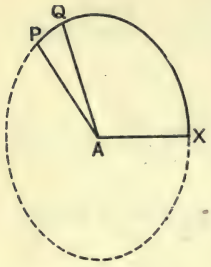

Fig. 79.

and this can be taken as the potential at $P$.

Note however this peculiarity, that the expression for the potential is not single-valued. If the positive angle $XAP = \theta$, the work done will be $\theta - 2\pi$ by a path (dotted in figure) that goes clockwise. It is $\theta + 2\pi$ by a counterclockwise path which makes a complete circuit of $A$ before proceeding to $X$; and its general value is $\theta + 2n\pi$, where $n$ is any integer positive or negative.

**Ex. 1.**—Find the field due to two equal, parallel and contrary currents $c$, acting in infinite straight lines.

Fig. 80 is drawn in a plane perpendicular to the two infinite straight lines. These therefore are seen as points $A$ and $B$, and the perpendiculars on them from $P$ are $PA$ and $PB$.

The fields due to the currents are $2c/AP$, $2c/BP$, respectively perpendicular to $AP$ and $BP$. Draw the circle $APB$, and let 0 be its centre and $PT$ tangent. Then

$$\angle TPA = \angle PBA$$

in alternate segment, and

$$TPB = 180° - PAB.$$

Hence resolving along $PT$ the component of field

$$= 2c \left\{ \frac{\sin TPB}{PB} - \frac{\sin TPA}{PA} \right\},$$

$$= 2c \left\{ \frac{\sin PAB}{PB} - \frac{\sin PBA}{PA} \right\} = 0.$$

Therefore the field is in direction $OP$, and its value

$$= 2c \left\{ \frac{\cos TPB}{PB} - \frac{\cos TPA}{PA} \right\},$$

$$= -\frac{2c}{PA.PB} \{PA \cos PAB + PB \cos PBA\},$$

$$= -\frac{2c.AB}{PA.PB}$$

by Trigonometry.

Since the field at every point $P$ is normal to the arc $APB$ of a circle through $A$ and $B$, such arcs are Equipotentials. The orthogonal system of circles are Lines of Force, such as the dotted circle through $P$ whose centre is $T$. They are a set of coaxial circles of which $A$ and $B$ are the limiting points, and whose radical axis bisects $AB$ perpendicularly.

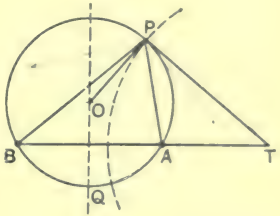


Fig. 80.

**Ex. 2.**—Find the potential at $P$ due to two equal and contrary currents as in the above example.

Let the arbitrary point of zero potential be chosen on $AB$ produced. Then if $A$ be the line of action of the current approaching, and $B$ of current receding, the potential due to $A$ can be taken as $-\angle TAB$, and due to $B$ as $+\angle TBP$; hence the sum

$$= TBP - TAP = \angle APB.$$

As this angle is constant for all points on same circular arc $APB$, this arc is an equipotential. It can be seen that the complementary arc of the same circle has potential $APB = \pi$.

**Ex. 3.**—Using the result of § 162, Ex. 5, and integrating with respect to $x$ between the values 0 and $x$, show that the potential of the polygonal current at distance $x$ from its centre, in direction perpendicular to its plane, is $2cn \tan^{-1}(\rho \sin\theta / 2a\rho) + \text{constant}$, where $n$ = number of sides, and $\tan\theta = x/R$.

**164. Ampere's Theorem.**—The law stated in § 160 is based on the experimental result, due to Ampere, that a plane circuit of small area has a field indistinguishable from that of a small magnet standing perpendicularly on the area. Of course the identity of fields only holds at points whose distances are considerable compared to the linear dimensions of the area or the magnet. In addition we assume (1) that the magnetic field is zero of two equal currents flowing contrary ways in parallel straight lines, when these lines approach and ultimately coincide; and (2) that the magnetic field is zero of two equal currents, one in a straight wire $AB$ and the other in an indented or finely coiled wire $BA'$ which lies around and always indefinitely near to $AB$. These assumptions were proved experimentally by Ampere.

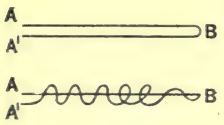

Fig. 81.

He used a wire bent back on itself, as $ABA'$ in first figure; $A$ and $A'$ being connected to a battery so that the same current flowed in $BA'$ as in $AB$. Then he made the modification indicated in second figure, winding and bending $BA'$. Either arrangement could be brought near a sensitive magnetometer without affecting it.

**165. Circuit of Finite Size.**—A circuit of any size and form can be supposed filled by a surface either plane or curved evenly without points or ridges. On this surface are drawn equal squares, which ultimately are infinitely small and of infinite number. Imagine these to be small circuits of insulated wire, each carrying the same current $c$ as the large circuit, the same way round. Wherever two squares are contiguous, the current in their touching sides are equal, contrary, and indefinitely close, and therefore have no magnetic effect, by (1), § 164. So the total magnetic effect of the current $c$ in the squares

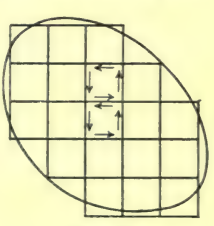
Fig. 82.

is identical with that of a current $c$ going round the outer border of the squares, and this border, in limit, is an indented line lying everywhere indefinitely close to the curve of the original circuit. It follows from (2), § 164, that this is equivalent to $c$ flowing in the actual curve. Hence in limit the magnetic effect of the circuit carrying current $c$ is identical with that of the indefinitely large number of equal squares described above, each carrying the same current $c$.

Each square is however (§ 164) equivalent magnetically to a small magnet perpendicular to its plane. The squares and their currents are all equal, and therefore so are the magnets. Hence the original circuit is equivalent to such a distribution of small equal magnets perpendicular to an even surface bounded by the circuit that equal numbers stand on equal areas.

**166. Magnetic Shells.**—A magnetic shell is a distribution of magnetised matter over a surface, such that the direction of magnetisation is everywhere normal to the surface. That is, a small portion of the surface has its magnetic moment in a direction perpendicular to the tangent plane. Magnetic shells are not found in nature and cannot easily be constructed artificially. They can be represented by supposing a large number of small simple magnets fixed normally to the surface.

A *uniform* shell is such that the magnetic moment per unit area is constant. This constant is the *strength* of the shell. Evidently the arrangement of magnets described in § 165 forms a uniform shell.

**167. Potential of a Uniform Shell.**—Let $s$ be its strength. Consider a small element of surface $\delta S$ at a point $P$, and let $O$ be the point at which the potential is required. The magnetic moment of $\delta S$ is in the direction of the normal $PN$ and has magnitude

$$M = s . \delta S.$$

The potential at $O$ (by § 153) is therefore

$$\delta V = \frac{s\delta S . \cos NPO}{PO^2}.$$

But $\delta S \cos NPO/PO^2 =$ the solid angle $\delta\Omega$ of the small cone whose vertex is $O$, and which stands on $\delta S$ as base, (cf. § 30). Hence the potential due to $\delta S$ is

$$\delta V = \epsilon . \delta\Omega,$$

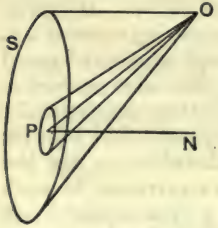


Fig. 83.

And if we add the potential at $O$ due to every element of the whole surface $S$, the potential

$$V = \epsilon\Omega,$$

where $\Omega$ is the whole solid angle subtended at $O$ by the periphery of $S$.

We have shown (§ 165) that the magnetic action of a current $c$ is identical with that of a uniform magnetic shell bounded by the circuit in which that current flows. Let us *define* the measure of a current as equal to the strength of the equivalent shell. We shall show later that this definition agrees with that of § 161.

Then the potential of a current circuit $= c\Omega =$ current $\times$ solid angle subtended by the circuit.

**Ex. 1.**—Find the potential at a point $P$ due to a given current $c$ which flows along an infinite straight line.

Let the straight line be called $AA'$ and let the plane of the figure be perpendicular to it and pass through $P$. Then the straight line $AA'$ is represented in the figure by a single point $A$. Let $AX$ be an arbitrary prime radius.

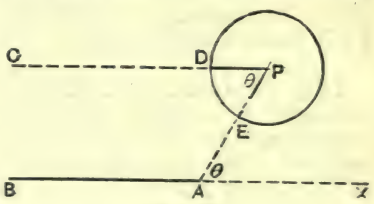


Fig. 84.

The method adopted applies only to complete circuits. Consequently the current $c$ in the infinite straight line $AA'$ must be replaced by one flowing in a very large circuit of which $AA'$ is one boundary. Let this circuit lie in the arbitrary plane $XAA'$, on the side remote from $X$. It is then equivalent to the infinite shell $AB$.

The straight lines joining $P$ to the perimeter of this shell generate the two planes $PAA'$, and $PC$ drawn through $A$

parallel to the plane $XAA'$. If a sphere of unit radius be drawn about $P$, these planes cut it in a lune $DPE$ of angle $\theta = XAP$. The area of such a lune is $2\theta$; consequently this is the solid angle.

The potential at $P$ is therefore $2c\theta$, if we suppose the circuit completed in the plane $XAA'$. As the plane is arbitrary, the absolute potential is indeterminate. But the potential difference of any two points is definite, and is the same wherever the prime radius $AX$ is taken.

This result agrees with that of § 163.

**Ex. 2.**—Find the potential due to a current in an infinite straight line and a return current in a parallel straight line.

Let the straight lines be $AA', BB'$, Fig. 79; and let the plane of the figure pass through $P$ and be perpendicular to them. The circuit can be supposed a rectangle bounded by $AA', BB'$, and two finite lines at an infinite distance. Hence the straight lines from $P$ to the boundary all lie in the two planes $PAA', PBB'$. These planes cut a sphere of unit radius about $P$ in a lune whose angle is $APB$. Hence the potential is $2.APB$.

The equipotentials are such loci that $\angle APB$ is constant. They are therefore cylinders (perpendicular to the plane of the figure) standing on circular arcs through $A$ and $B$. Cf. Ex. 2, § 163.

**Ex. 3.**—If $r$ be the radius of a circular coil of one turn carrying current $c$, and $z$ be distance of a point $O$ on the axis from the centre of the circle, show that the circle subtends at $O$ a solid angle $2\pi(1 - z/\sqrt{z^2 + r^2})$. Deduce the potential at $O$. By differentiating the potential with respect to $z$ show that the field at $O$ is $2\pi r^2 c/(z^2 + r^2)^{3/2}$.

**Ex. 4.**—A current flows once round a square whose side is $a$. Find the potential at the centre of the cube of which the square is one face.

**Ex. 5.**—$O$ is the centre of a polyhedron of $n$ faces, all equal. If a current $c$ flows around one face, show that the potential at $O$ is $4\pi c/n$.

**Ex. 6.** Find the potential due to a current flowing in a regular polygon at any point equidistant from all the angles of the polygon.

**168. Cyclic Constants.**—Consider a magnetic shell (see Fig. 85) in form of a plane disc. When $O$ is at $A$, indefinitely near it in front, *i.e.* on the $N$ side, the solid angle subtended can be taken as $2\pi$, for this is half the

total solid angle subtended by all directions round a point. The potential is $2\pi s$, if $s$ be the strength. As $C$ recedes along any path $ABC \ldots$, it would be repelled if it were a unit north pole; consequently the potential falls from its value at $A$. In figure, drawn nearly to scale for a circular disc, the angles subtended at $B$, $C$, . . . . are $\pi$, $\frac{1}{2}\pi$, etc., as marked, and the potentials are therefore $\pi s$, $\frac{1}{2}\pi s$, . . . . At $D$ in plane of disc the solid angle and potential are zero; but, since the disc is equivalent to a small magnet pointing in direction $GA$, the field at $D$ is parallel to $AG$ and the potential gets still less than zero as we travel onwards. It is negative and takes the values $-\frac{1}{2}\pi s$, $-\pi s$, $-2\pi s$ at $E$, $F$, $G$. Consequently the difference of potentials at $A$ and $G$ is $4\pi s$ for the journey $ABC \ldots G$.

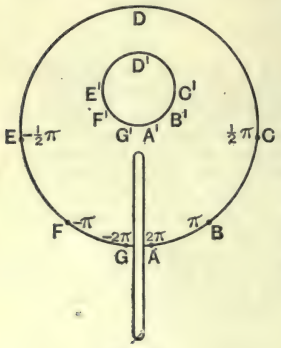

Fig. 85.

Now replace the disc by its equivalent current, so that $s = c$. It becomes possible for $O$ to pass direct from $G$ to $A$; and since the distance is indefinitely small, no work is done, that is $G$ and $A$ are at the same potential for the infinitesimal journey $AG$. But $4\pi c$ of work is done by magnetic forces on unit pole travelling from $A$, by path $ABCDEFGA$, back to $A$ again; so that a finite quantity of work is done in a closed cycle beginning and ending at the same point. When any summation or integration gives a finite value when taken round a closed cycle, this value is called the Cyclic Constant.

It is of course evident that if a unit pole had travelled round the other way, $AGFEDCB$, the work would have had the contrary sign. Had it travelled round $ABC \ldots A$ an integral number of times, say $n$, the work would have been $4\pi nc$. The work done in going round a closed path $A'B' \ldots G'A'$, which does not loop through the circuit, is of course zero.

**169. Magnetomotive Force (M.M.F.).** The work which is done by the magnetic forces on a unit $N$ pole as it travels completely once round a circuit is called the magnetomotive force in that circuit. This work is reversible, for an equal amount of work must be supplied in order to carry the $N$ pole around the circuit in the contrary direction.

There is evidently a close analogy between M.M.F. and E.M.F. (§ 102). The most obvious difference is this. We usually meet with E.M.F. in a circuit around which electricity is flowing, but there is nothing flowing around a circuit in virtue of its M.M.F. Hence E.M.F. can be defined as the work per unit charge making the circuit, but M.M.F. is the work that would be done on unit $N$ pole if it could be imagined to make the magnetic circuit.

By last paragraph, the M.M.F. in a circuit which surrounds once a current $c$ is $4\pi c$.

**170. Webers and Ampere Turns.**—In §§ 161-169 the magnitude of the current was taken equal to that of the equivalent magnetic shell. That is, the current $c$ is in absolute units. The M.M.F., $4\pi c$, is then in webers; a weber denoting the M.M.F. of such a circuit that one erg is the work done on a unit $N$ pole travelling round it once. Engineers prefer to use another unit called the Ampere Turn, which is the M.M.F. of a circuit which once embraces a current of 1 ampere. Since an ampere = 1·10 absolute units, therefore an ampere turn = $4\pi \times 1 \cdot 10$ = 1·257 webers.

**171. Solenoids.**—A solenoid is a helical coil of wire, such as would be constructed by winding wire evenly round a long straight circular ruler. The wire may be in one or more layers.

Very approximately the effect of a solenoid traversed by a current is the same as that of a number of circular wires of the same radius, whose centres are arranged at equal distances along a common axis ($AD$ in Fig. 86) perpendicular to their plane.

Let $a$ = mean radius of the winding = the radius of

any one of these circles. Let $n$ = the number of turns, or equivalent circles. Let $l$ = the length of the cylinder occupied by the winding. Let $C$ be the current in amperes; $C/10$ or $c$ = current in absolute units.

Each circle is equivalent to a magnetic shell of strength $c$ or $C/10$ and area $\pi a^2$, and therefore of magnetic moment $\pi a^2 c$. Hence the whole magnetic moment of the solenoid is $\pi a^2 nc$. This is altogether independent of $l$. The formula holds for a short wide cylindrical winding, like the coil of a tangent galvanometer, or a long narrow one.

The M.M.F. is $4\pi nc$ or $4\pi nC/10$. This too is independent of $l$. It can also be expressed as $nC$ ampere-turns.

**172. Field in a Solenoid.**—The field outside, at a sensible distance from the ends, is very small compared to that within the hollow. The field inside is sensibly uniform*; let it be called $F$. Hence, nearly the whole work on a unit pole completing the circuit $ABDA$ (Fig. 86) is done inside, and $= Fl$.

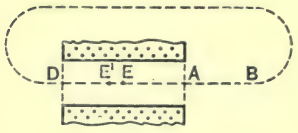

Fig. 86.

Therefore

$$Fl = 4\pi nC/10,$$

$$F = 4\pi nC/10l,$$

or

$$= 4\pi NC/10,$$

where $N$ is the number of "turns per unit length."

It is evident that the field $F$ at $E$ can be considered as the sum of fields due to parts $EA$ and $ED$ of solenoid. These parts (supposed long) have equal effect, consequently each exerts a field $\frac{1}{2}F$. It follows that the field of a solenoid precisely at its end, in the plane of the last turn, is $\frac{1}{2}F$, *i.e.* half as great as the field in the interior.

**173. Field of a Solenoid. Exact Treatment.**—Each turn is equivalent to a plane circular magnetic shell of area $\pi a^2$ and uniform strength $= c = C/10$, so that its

* See precise working which follows lower down.

magnetic moment is $\pi a^2 c$. There are $n$ turns in a length $l$; so if $l/n$ be small we can consider each magnetic shell as a magnetic disc of length $l/n$ and pole strength $m = \pi a^2 cn/l$, its poles being its flat ends. These $n$ discs are successively in contact, the $S$ pole of each annulling the equal $N$ pole of the next, so that only the poles at $A$ and $D$ are left uncompensated.

The field at a distant point $B$ is nearly $m/BA^2 - m/BD^2$ if $a$ be small. This

$$= \frac{\pi a^2 cn}{l} \left\{ \frac{1}{BA^2} - \frac{1}{BD^2} \right\}.$$

The field at a point quite close to $A$, but just outside, can be divided into two parts. The disc pole at $A$ has uniform density $\sigma = m/\pi a^2 = cn/l$; consequently it exerts a force $2\pi\sigma$ (§ 34) $= 2\pi cn/l$. The pole at $D$ exerts a force $-m/AD^2 = -\pi a^2 cn/l^2$. For a long solenoid we can neglect this term and put the field $= 2\pi cn/l$.

The field at $E$ inside the solenoid is the sum of the fields due to the two solenoids $EA$ and $DE$. If these can both be treated as long, the field of each is $2\pi cn/l$, and the total field at $E$ is $4\pi cn/l$.

If $B$ be on the axis of the solenoid, at a distance $z$ from $A$, the field due to the flat disc $\pi a^2$ at $A$ of surface density $\sigma$ can be shown to be

$$2\pi\sigma \{1 - z/\sqrt{a^2 + z^2}\}.$$

If

$$z' = DB,$$

the field due to $D$

$$= -2\pi\sigma \{1 - z'/\sqrt{a^2 + z'^2}\}.$$

Hence accurately the field at $B$

$$= 2\pi\sigma \left\{ \frac{z'}{\sqrt{a^2 + z'^2}} - \frac{z}{\sqrt{a^2 + z^2}} \right\}.$$

If $B$ be at $A$, this

$$= \frac{2\pi\sigma l}{\sqrt{a^2 + l^2}}.$$

If $z$ and $z'$ be the distances of $E$ from $A$ and $D$, the field at $E$ = sum of the fields just outside two solenoids of length $z$ and $z'$,

$$2\pi\sigma \left\{ \frac{z'}{\sqrt{a^2 + z'^2}} + \frac{z}{\sqrt{a^2 + z^2}} + 2 \right\}.$$

Of course $\sigma = cn/l$.

**Ex. 1.**—Find the field in the interior of a solenoid 10 cm. long, 1 cm. radius, of 500 turns, and carrying a current of half an ampere.

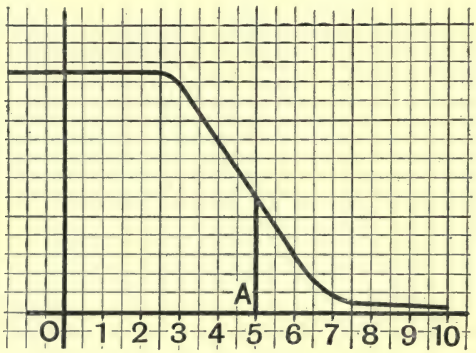


Fig. 87.

The radius is immaterial, if we regard it as small. The field = $4\pi Nc$, $c = 1/20$, and $N = 500/10 = 50$, therefore the field = $10\pi$ = 31·4. If we use the strict result given by the Calculus, the field at the centre

$$= 4\pi Nc \times 5/\sqrt{5^2 + 1^2} = 30{\cdot}8.$$

The difference is relatively large, about 2% of the whole, because the diameter of the solenoid is not very small compared to its length.

**Ex. 2.**—For the same solenoid, find the field at the centre of one end.

The approximate formula gives $2\pi Nc = 15{\cdot}7$. The strict formula gives 15·6.

**Ex. 3.**—For the same solenoid find the field at a point on the axis, 5 cm. from the end.

The approximate formula treats the solenoid as a magnet whose pole strength is $N\pi a^2 c = 50\pi/20 = 5\pi/2$. The poles are at distances 5 and 15 from the point considered, therefore the field is

$$\frac{5\pi}{2}\,(1/5^2 - 1/15^2) = \pi/10 - \pi/90 = \cdot 279.$$

The exact formula gives ·274.

**Ex. 4.**—Draw a curve showing the relation of field to position on axis for the above solenoid.

The results found in Ex. 1—3 above are utilised in the graph shown ; along with other results found similarly.

**Ex. 5.**—Find the cyclic constant of the above coil with the current $\frac{1}{2}$ ampere.

There are 500 turns, so the constant is 250 ampere turns, or $4\pi \,.\, 500/20 = 100\pi = 314$ absolute units.

**Ex. 6.**—An anchor ring is wound uniformly with 1000 turns of wire. The radius of its section is ·6 cm. ; the radius of the ring itself (from the centre of the ring to the centre of its section) is 8 cm. Find the field within and outside the windings, and the M.M.F. in the core, for a current of 1 ampere.

Obviously the M.M.F. is 1000 ampere turns, or $4\pi \times 1000 \times 1/10 = 1257$ webers. Let $F$ be the field at distance $r$ from the axis of ring. A unit pole travelling round the axis at this distance describes a distance $4\pi r$ cm. ; hence the work done on it

$$= 4\pi r \,.\, F = 4\pi \times 1000/10.$$

Hence

$$F = 100/r.$$

Its mean value can be taken as $100/8 = 12{\cdot}5$ gausses, very nearly.

This wound ring can be regarded as a set of 1000 magnetic shells each equivalent to one turn of wire. Their thickness is what we like (provided it is small), and we can suppose the shells to have a mean thickness $2\pi \,.\, 8/1000$ and be bounded by planes which are not parallel but intersect at the axis of the ring. They will then fit close together, and the + face of each will exactly cancel (for external points) the − face of the next. Hence there will be no magnetic field at all exerted at outside points. The whole of the lines of magnetic flux are internal, and the arrangement is not magnetic except in its circular core.

**Ex. 7.**—Find the magnetic moment of a solenoid of length 20 cm., radius ·6 cm., number of turns 400, current ·4 amp. Find also the difference of magnetic potential (approximate) between its ends ; and the field (1) in the interior, (2) exactly at the centre of an end face, (3) 50 cm. from the centre in a direction along, and at right angles to, the axis of the coil.

**174. Best Shape of a Section of a Galvanometer Coil.**—Let $O$ be the point at which the field is to be measured; $PQ$ the bounding curve of the section. Let $OX$ be the axis of symmetry of the coil; the resultant field will be along $OX$. The problem is to find the form of the curve $PQ$ which produces the largest field along $OX$ with a given quantity of wire.

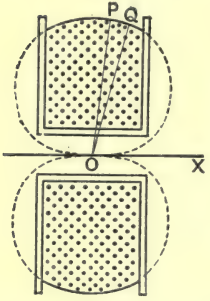

Fig. 88.

A length $l$ of wire at $P$ produces field $cl/OP^2$ at $O$ perpendicular to $OP$, and therefore produces component $cl \sin XOP/OP^2$ along the axis. If the coil were wound in such a way that this length were taken off at $P$ and put on at $Q$, it would produce $cl \sin XOQ/OQ^2$. The most economical winding has these two expressions equal; since otherwise we could get a larger field by transferring wire from one place to the other. So the form of the bounding curve is such that $\sin XOP/OP^2 =$ constant. The curve has the polar equation

$$r^2 = a^2 \sin \theta.$$

**175. Moving Coil Galvanometers.**—A coil of $n$ parallel turns carrying current $c$ (absolute) is equivalent to $n$ plane parallel coils carrying the same current, each of one turn. Let $A$ be their total area. As their planes are parallel, their magnetic moments are additive and the sum is $Ac$. If the coil is in a magnetic field $H$, whose direction makes angle $\phi$ with the normal to the planes, it is acted on by a couple

$$HAc \sin \phi.$$

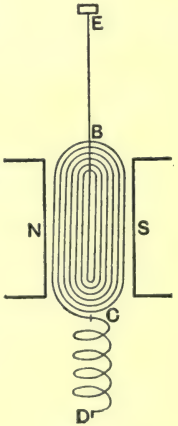

Fig. 89.

Let the coil be suspended by a vertical torsion wire so

that the planes of its coils are vertical and contain a horizontal magnetic field $H$ when the wire is untwisted. Let $L$ be the torsion constant of the wire, so that $L\theta$ is the couple exerted by the wire when its twist is $\theta$. In that position $\phi = 90° - \theta$, hence the condition for equilibrium is

$$HAc \cos \theta = L\theta.$$

If the coil be suspended bifilarly, and $L$ be the constant of the bifilar suspension, the couple is $L \sin \theta$ and the equilibrium condition is

$$HAc = L \tan \theta.$$

Of course $H$ can either be the earth's horizontal field or a field produced by permanent magnets or electric currents.

If the suspension be bifilar, one wire can bring the current to the coil and the other take it away. If the suspension be torsional, the current can be brought by the torsion wire and carried away by a vertical wire underneath dipping into mercury, or by a fine wire twisted into a very feeble spiral spring. The figure shows diagrammatically the coil $BC$ of an Ayrton-Mather galvanometer. $EB$ is the torsion wire, $CD$ a spiral, $E$ and $D$ are the electrodes, $N$ and $S$ are poles of a strong permanent magnet.

**Ex. 1.**—A circular coil of 100 turns, radius 5, hangs in the magnetic meridian by a torsion wire which would be twisted 6° by a couple of magnitude 200 absolute units. Find through what angle it turns if a current $c$ (absolute units) passes.

If couple 200 produces a twist of 6°, $= \pi/30$ radians, then

$$L = 6000/\pi.$$

Since there are 100 turns of area $25\pi$,

$$A = 2500\pi.$$

Hence the equation

$$HAc \cos \theta = L\theta$$

becomes

$$\cdot 186 \times 2500\pi c \cos \theta = 6000\, \theta/\pi,$$

giving

$$\theta/\cos \theta = \cdot 765c.$$

Fig. 90.

It is usually better to find the number of degrees in $\theta$: let it be $D$. Then

$$\theta/\pi = D/180,$$

and we obtain $D/\cos D = \cdot 0765 \times 180c/\pi = 4{\cdot}38c$.

**Ex. 2.**—Give numerical results for above coil, if the current be one-tenth ampere. If the deflexion is read by a mirror and scale, the scale being 1 metre distant, find the scale reading.

Here $$c = 1/100,$$
$$\therefore\ D/\cos D = \cdot0438.$$

As the angle is very small, we take cos $D = 1$. So $D = \cdot0438°$.
Also $\theta = \cdot00765$ radian. This is the angle rotated by the mirror, therefore the reflected ray rotates $2\theta = \cdot0143$. The arc subtended by $2\theta$ at 1 metre distance is $2\theta \times 100 = 1\cdot43$ cm. ; hence $1\cdot43$ is the scale reading.

**Ex. 3.**—A circular coil is suspended bifilarly in the earth's field, and turns through angle $\theta$ when a certain current passes. If however it be not allowed to turn, it will deflect a compass needle, placed at its centre, through an angle $\phi$. Show that we have sufficient data to find both the current and the earth's horizontal field. We have

$$A = n\pi a^2,$$
$$\therefore\ Hc = L \tan\theta/n\pi a^2.$$

But by the formula for a tangent galvanometer

$$c = aH \tan\phi/2\pi n.$$

Hence

$$H^2 = 2L \tan\theta/a^3 \tan\phi$$

(a formula which does not contain $n$), and

$$c^2 = L \tan\theta \tan\phi/2\pi^2 an.$$

**Ex. 4.**—A torsion wire is twisted 1° per couple of 100 dyne cm. It supports a rectangular coil of 250 turns $1 \times 3$ cm., in a magnetic field, between the poles of a fixed magnet, of strength 120. Calculate the current indicated per 1° deflection.

**Ex. 5.**—If the torsion wire in the coil of Ex. 4 is replaced by another one, and it be observed that the period of oscillation of the suspended coil is halved, how much will this affect the sensitiveness of the galvanometer, *i.e.* the current indicated per 1°?

**Ex. 6.**—A coil of total area 50 sq. cm. and weight ·5 gm. is supported bifilarly by two perfectly flexible wires, 2 mm. apart and 5 cm. long, which act as leads to the current. Calculate the current for a deflection of 1°, the magnetic field being that of the earth (·186).

**176. Direct Reading Galvanometers. Ammeters and Voltmeters.**—Every galvanometer measures primarily the current through itself ; and secondarily the P.D.,

which = current × galvanometer resistance, between its terminals. It can therefore be graduated either in current units or in units of P.D. If used to measure currents and, graduated in amperes, it is an Amperemeter or Ammeter; similarly a Milliammeter is graduated in thousandth parts of an ampere, and a Microammeter in millionths. If used to measure P.D. and graduated in volts it is a voltmeter; and we can also have millivoltmeters, and microvoltmeters.

For an experiment we may want to know the current supplied to an apparatus (*e.g.* an incandescent lamp, or a dynamo) and the P.D. between the poles of the apparatus. In the first case we use an ammeter which is placed in series with the apparatus and measures its actual current. But it is expedient that the presence or absence of the ammeter shall not appreciably affect the current measured; therefore the ammeter ought to be of small resistance compared to the pieces of apparatus with which it is generally used. In the second case a voltmeter is placed in parallel with the apparatus, and measures the actual P.D. between its terminals. But it is expedient that the presence or absence of the voltmeter shall not appreciably affect the P.D. measured; consequently the voltmeter should have a large resistance compared to the apparatus.

Note that if a voltmeter of very high resistance be shunted with a 1 ohm coil it becomes an ammeter. For if it be put in series with a circuit, practically the whole current goes through the 1 ohm; and every ampere of current produces 1 volt P.D. between its terminals. In the same way a milliammeter becomes a voltmeter if its resistance be made up to 1000 ohms by putting a box in series with it.

**177. Dynamometers and Current Balances.**—These are instruments for measuring the force or couple exerted by one coil upon another situated in a standard position. They thus estimate the product of the two currents.

If the two coils are in series, the two currents are the same; let its value be $c$. A dynamometer then measures

$c^2$ directly. The reading is unaltered if the current be reversed. If the current be an alternating or fluctuating one, the value of $c^2$ is continually changing, but usually the changes are so rapid that a measurement can be made which represents the average value of $c^2$. This is important if $c$ varies harmonically. Then (§ 181) the average of $c^2$ is half the square of the maximum current.

Another way of connecting is to put one of the two coils, which has low resistance, in series with an apparatus, so that it is traversed by the whole current supplied; and to put the other coil, which has high resistance, in parallel with the apparatus, so that its current represents the voltage between the terminals. Then the dynamometer measures the product of voltage (P.D.) and amperage (current); *i.e.* it gives the wattage, or number of joules of energy supplied per second. It thus measures Power directly, and that is the reason for the name Dynamometer.

**178. The Weber Electro-Dynamometer.** — The instrument has two coils. The large fixed one is like that of a tangent galvanometer, with its plane vertical and in the magnetic meridian. The small one hangs at the centre of the large one by a torsion wire depending from a torsion head. This wire is untwisted, and the torsion head reads zero, when the axis of the small coil is in the plane of the large one. When currents flow in the two coils, the small one is deflected and has to be brought back to its standard position by rotating the torsion head. The current in the small coil enters by the torsion wire, and leaves by a fine wire spiral underneath, or by a wire dipping into mercury, or any arrangement which minimises torsion.

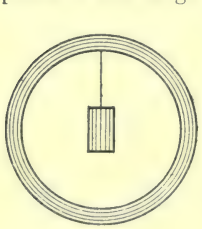

Fig. 91.

Let $a$, $b$ be radii of large and small coil; $m$, $n$ be numbers of turns; $x$, $y$ be currents in absolute units.

The field at the centre of the large coil is $2\pi mx/a$

(§ 161, Ex. 1). The magnetic moment of small coil is $\pi nb^2y$. If $\theta$ be the torsion angle measured on the torsion head, and $L$ a constant for the torsion wire, the couple is $L\theta$. Hence

$$L\theta = \pi nb^2y \times 2\pi mx/a = 2\pi^2b^2mnxy/a,$$

so that $xy$ is measured at once by the angle $\theta$.

*Note.*—The earth's field $H$ exerts no couple on the coil, for all measurements are made when its axis is pointing northward.

The British Association Electrodynamometer is of this type with certain modifications. The large coil is replaced by a pair of coils arranged as in a Helmholtz galvanometer [§ 161, Ex. 8]; radii 25 cm. and distance apart 25 cm. The number of turns used is any multiple of 30 up to 450. The suspension is bifilar, and its sensitiveness is adjustable by altering the distance apart of the wires at top.

**Example.**—If the fixed coil be 50 turns of radius 14, and the suspended coil be 700 turns of radius 1, both being in series, calculate the couple exerted on the suspended coil, kept in standard position, by a current of 1 ampere.

**179. The Joule Current Balance.**—The fixed and moving coils are parallel. When like currents flow, they act as if they were two magnets whose dissimilar poles are opposed; they therefore attract each other. The weights needed to hold the movable coil in a standard position are proportional to the product of the currents.

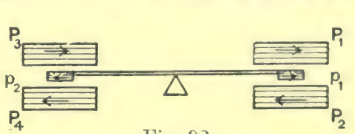

Fig. 92.

An improved form is the Kelvin current balance.

The four coils $P$ represent the fixed system. Let each have radius $a$, and number of turns $m$; and let $x$ be their current.

The two coils $p$ represent the movable system. They are smaller; if their radii be $b$ and number of turns $n$ and current $y$ they can each be regarded as a magnet of moment $M = n\pi b^2y$.

Let the current $x$ be the same way round in $P_1$ and $P_3$, and the contrary way in $P_2$ and $P_4$, as indicated by the

arrows. Let $y$ be one way in $p_1$ and the other in $p_2$. Then $P_1$ and $P_2$ unite to exert a force, supposed upwards, on $p_1$; while $P_3$ and $P_4$ exert an equal contrary force on $p_2$. The earth's field exerts no couple on $p_1p_2$, for the currents in it are opposite ways. The couple exerted by $P_1P_2P_3P_4$ on $p_1p_2$ (in the standard position) is found at once from the formula in Ex. 10, § 161, by regarding each small coil as replaced by a magnet of the same moment.

The couple depressing $p_1$ is balanced by weights in such a way as to bring $p_1p_2$ to its standard position; and it is evident that the required gravitational couple is a calculable multiple of $xy$. The weights used are generally sliders which can be placed on the bar connecting the movable coils.

These instruments can be used for the direct measurement of a single current. Putting the two coils in series we have $x = y$ and therefore the quantity measured is $x^2$.

They can also be used as Wattmeters, as above (§ 177).

# CHAPTER X.

## BALLISTIC DISCHARGE AND MEASUREMENT OF CAPACITY.

**180. Fluxional Notation.**—Let $x$ be any varying quantity whatsoever. Let $t$ be the time, reckoned from any chosen epoch. Let the value $x$ occur at time $t$; then we denote the value at time $t + \delta t$ by $x + \delta x$. The quantity alters by $\delta x$ in the interval $\delta t$.

The average rate at which $x$ alters for this interval is, of course, $\delta x/\delta t$.

If the interval $\delta t$ is taken very small, and ultimately zero, then $\delta x$ also is small and ultimately vanishing, for (in any natural process) no quantity changes with absolute suddenness. Ordinarily the fraction $\delta x/\delta t$ approaches a limit. This limit is denoted by $dx/dt$ or by $\dot{x}$. Instead of writing

$$\frac{dx}{dt} = \dot{x},$$

it is permissible to write

$$dx = \dot{x}dt$$

as an abbreviation, for $\delta x$ approximates to the value $\dot{x}\delta t$ when $\delta t$, and therefore $\delta x$, are small and ultimately vanishing.

In words, $\dot{x}$ denotes the "rate of change of $x$." If $x$ be the distance which a point has traversed, $\dot{x}$ will be the velocity of that point. One uses $\ddot{x}$ for $d^2x/dt^2$ to denote the rate at which $\dot{x}$ alters. If therefore $x$ be velocity, $\ddot{x}$ will be acceleration. Another important case is that of angular motion. If $\theta$ be the angle already described at time $t$, $\dot{\theta}$ is the angular velocity and $\ddot{\theta}$ the angular acceleration. If V

be the quantity of water in a reservoir, $\dot{V}$ will be the rate at which $V$ is increasing, or the current flowing into the reservoir. If $V$ be negative, the water is actually flowing out. Similarly if $q$ = the electric charge in a suitable receptacle (an accumulator or condenser), $\dot{q}$ will be the current inwards, or $-\dot{q}$ the current outwards. Any number of other instances might be given.

**181. Mathematical Formulae. Harmonic Change.**—If $\ddot{x} \propto -x$, let $\ddot{x} + n^2x = o$ where $n$ is a constant. This is the equation of simple harmonic motion. The general solution (which the mathematical student can verify by differentiating) is

$$x = a \sin(nt + E),$$

$a$ and $E$ being constants.

The values of $x$ recur when $t$ is increased by $2\pi/n$. This therefore is the period $T$. The rate of change of $x$ is

$$\dot{x} = an \cos(nt + E).$$

The maximum value of $x$ is called the Amplitude. It occurs when $nt + E = \frac{\pi}{2}$; and is $= a$. The maximum value of $\dot{x}$ is $an$, and therefore is equal to $2\pi a/T$. The constant $E$ is called the Epoch. It $= o$ if we reckon $t$ from the instant when $x$ vanishes.

The average value (algebraic) of $x$ is zero, for it is as often positive as negative. But

$$x^2 = a^2 \sin^2(nt + E) = \tfrac{1}{2}a^2 - \tfrac{1}{2}a^2 \cos 2(nt + E).$$

The average of the second term is zero, therefore $\frac{1}{2}a^2$ is the average of $x^2$. The average square of a quantity varying harmonically is half the maximum square.

**182. Logarithmic Decrement.**—If $\dot{x} \propto -x$, let $\dot{x} + \lambda x = o$ or $\dot{x}/x = -\lambda$, where $\lambda$ is constant. Now $\dot{x}/x$ or $\frac{1}{x}\frac{dx}{dt}$ is the rate of change of the Napierian Logarithm of $x$.

Hence

$$\frac{d}{dt}\log x = -\lambda;$$

therefore $\log x$ diminishes at the constant rate $\lambda$.

The name Logarithmic Decrement is given to $\lambda$. If the initial value (when $t = o$) of $x$ be $x_0$,

$$\log x_0 - \log x = \lambda t,$$
$$\lambda = (\log x_0 - \log x)/t,$$
$$x_0/x = e^{\lambda t},$$
$$\therefore\ x = x_0 e^{-\lambda t}.$$

The above logarithms are to base $e$. They can be got by multiplying logs. to base 10 by 2·3026; hence

$$\lambda = (\log_{10} x_0 - \log_{10} x)/2{\cdot}3026t.$$

**183. Resisted Harmonic Motion.**—Consider the equation $\ddot{x} + 2\lambda\dot{x} + n^2x = o$.

First assume that the solution is $x = ae^{-kt}$, $a$ and $k$ being constants. Then $\dot{x} = -kx$, and $\ddot{x} = k^2x$, consequently

$$k^2 - 2k\lambda + n^2 = o,$$

giving

$$k = \lambda \pm \sqrt{\lambda^2 - n^2}.$$

This solution is perfectly suitable if $\lambda^2 - n^2$ be positive. There are then two values of $k$, say $k_1$ and $k_2$; and $ae^{-k_1t}$, $be^{-k_2t}$ are independent solutions. It can be verified that

$$x = ae^{-k_1t} + be^{-k_2t}$$

is a general solution containing two arbitrary constants $a$ and $b$.

The above solution fails if $\lambda^2 - n^2$ be negative. This is the more interesting case.

To solve it, put

$$n^2 = \lambda^2 + m^2;$$

then choose a new variable $y$ such that

$$x = ye^{-\lambda t}.$$

By differentiation,

$$\dot{x} = (\dot{y} - \lambda y)e^{-\lambda t},$$

and

$$\ddot{x} = (\ddot{y} - 2\lambda\dot{y} + \lambda^2 y)e^{-\lambda t}.$$

Substitute these in the differential equation. After dividing by $e^{-\lambda t}$ we get

$$\ddot{y} + m^2 y = o,$$

of which the solution (§ 181) is

$$y = a \sin (mt + E),$$

therefore

$$x = ae^{-\lambda t} \sin (mt + E).$$

This can be regarded as a Simple Harmonic Motion whose amplitude is not constant, but is always $= ae^{-\lambda t}$.

Hence $\lambda$ is the Logarithmic Decrement of the Amplitude.

**184. Free Oscillations of a Galvanometer Needle.**—Consider a needle suspended without friction. If it be displaced through a small angle $\theta$, its suspension exerts on it a couple tending to replace it, and ultimately proportional to $\theta$. This couple may therefore be denoted by $-L\theta$, where $L$ is approximately a constant.

Employing Newton's Fluxional Notation, we denote the angular velocity by $\dot{\theta}$ and the angular acceleration by $\ddot{\theta}$. The same quantities are denoted by $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$, $t$ being the time.

If $K$ be the moment of inertia of the needle, the acting couple must equal the product of $K$ and the angular acceleration. If therefore the needle be simply swinging under no disturbing force, the equation of motion is

$$K\ddot{\theta} = -L\theta,$$

or

$$K\frac{d^2\theta}{dt^2} + L\theta = o,$$

and the motion is Simple Harmonic (by § 181, putting $L/K = n^2$), of period

$$T = 2\pi\sqrt{K/L}.$$

The full solution of the equation of motion can be written

$$\theta = a \sin t\sqrt{L/K},$$

where evidently $t$ is the time reckoned from that instant at which $\theta = o$. The greatest value of $\theta$ is $\theta = a$ it occurs when

$$t\sqrt{L/K} = \frac{\pi}{2},$$

$$t = \frac{\pi}{2}\sqrt{K/L} = \tfrac{1}{4}T.$$

The value of $\dot{\theta}$ or $\dfrac{d\theta}{dt}$ is

$$\dot{\theta} = a\sqrt{L/K}\cos t\sqrt{L/K};$$

and its greatest value is when $t = o$, and is

$$\dot{\theta} = \omega_o = a\sqrt{L/K} = 2\pi a/T.$$

**185. Transient Currents.**—A current in a galvanometer is called transient if its total duration is a time small compared to the period of oscillation of the galvanometer needle. It is evident, of course, that a current may be transient in one galvanometer and not transient in another.

In most forms of galvanometer (§ 175 and Exs. on § 161) a couple acts on the needle which is simply proportional to the current when the deflection $\theta$ is small. Let $Ac$ be the couple for current $c$ (absolute units).

If $c$ be a steady current, equilibrium is reached for such a deflection $\theta$ that this couple balances that due to the suspension, so that $Ac = L\theta$ (§ 180). Let $C$ be the ordinary galvanometer constant, *i.e.* the current indicated per unit deflection. Then

$$c = C\theta,\ AC = L,\ C = L/A.$$

If $c$ be the average value of a transient current of short duration $t$, $Ac$ is the average couple, and $Ac/K$ is the average angular acceleration for time $t$. The total angular velocity given in this time is $Act/K$. If $q = ct$, $q$ is the total electric charge carried by the transient current through the galvanometer. It gives the needle an instantaneous angular velocity $Aq/K$.

Let the needle be at rest in its equilibrium position

when this happens. It will be set in motion with the above angular velocity, and will start to oscillate freely in its natural period, with a maximum angular displacement, $a$ say. The maximum angular velocity is the value of $\dot{\theta}$ when $\theta = o$; it is $2\pi a/T$ (§ 184).

Hence

$$Aq/K = 2\pi a/T.$$

It follows that $a$ can be used to measure $q$. The galvanometer is then said to be used **Ballistically,** and the **Ballistic Constant** $Q$ is the value of $q$ which would correspond to $a = 1$; so that

$$q = Qa\,;\ Q = 2\pi K/AT.$$

Compare this with the Ordinary Constant. By division,

$$Q/C = 2\pi K/LT.$$

But

$$T = 2\pi\sqrt{L/K},\ K/L = 4\pi^2/T^2\,;$$

hence finally

$$Q/C = T/2\pi.$$

This very simple relation holds also in coulombs and amperes. It enables the Ballistic Constant to be found at once when the Ordinary Constant has been determined.

**Ex. 1.**—The complete period of swing of the undisturbed mirror of a galvanometer is 3 seconds. A current of one microampere produces deflection of the reflected light spot through 8 divisions of the scale. What charge is indicated when the galvanometer is used ballistically and the spot swings to maximum distance 13 divisions from central reading?

Measure $c$ and $q$ as current and charge per 1 cm. deflection of spot of light. Since 8 cm. indicates $10^{-6}$ amps., $c = 10^{-6}/8 = 1{\cdot}25 \times 10^{-7}$. Since $q = cT/2\pi$, $q = 3 \times 1{\cdot}25/2\pi \times 10^{-7}$. The charge required is $13q = 7{\cdot}76 \times 10^{-7}$ or ·776 microcoulomb.

**Ex. 2.**—Find the ratio of the ballistic constant in coulombs to the current constant in amperes if the period of oscillation is 1 second.

**186. Frictional Resistance to Galvanometer Needle.**—If the resistance be due either to the air, or to a liquid, or to induced currents, it is proportional to the

angular velocity. A couple acts on the needle proportional to $\dot{\theta}$ and opposing the motion. Call it $N\dot{\theta}$; we therefore get the equation

$$K\ddot{\theta} = -N\dot{\theta} - L\theta,$$

or

$$K\ddot{\theta} + N\dot{\theta} + L\theta = o.$$

This equation is of the form considered in § 183, which may here be written (replacing $x$ by $\theta$)

$$\ddot{\theta} + 2\lambda\dot{\theta} + n^2\theta = o.$$

In any galvanometer suitable for ballistic work, $\lambda$ is much smaller than $n$, therefore $\lambda^2 - n^2$ is negative, and the solution is

$$\theta = ae^{-\lambda t}\sin(mt + E)$$

where

$$m^2 = n^2 - \lambda^2.$$

We can ordinarily neglect $\lambda^2$ (but not $\lambda$), and therefore write $m = n$.

Comparing the equations, we see that the logarithmic decrement is

$$\lambda = N/2K.$$

If there were no friction, $N = o$ and $\lambda = o$; and we should have

$$\text{Period } T = 2\pi/n = 2\pi\sqrt{K/L}.$$

The actual period of the term $\sin(mt + E)$ is however

$$T' = 2\pi/m = 2\pi/\sqrt{n^2 - \lambda^2}$$
$$= 2\pi\Big/\sqrt{\left(\frac{L}{K} - \frac{N^2}{4K^2}\right)}.$$

Thus the observed period $T'$ differs from $T$, the corrected period. But if $\lambda^2$ be negligible, $T = T'$.

**187. Apparent and Corrected Amplitude.**— Reckon the time from the instant when $\theta = o$. Neglect $\lambda^2$ and therefore put $T = T'$, $m = n$. Then

$$\theta = ae^{-\lambda t}\sin nt.$$

The first outward swing terminates when

$$nt = \frac{\pi}{2},^* \; t = \pi/2n = T/4,$$

hence

$$\theta = a_1 = ae^{-\lambda T/4}, \; a/a_1 = e^{\lambda T/4}.$$

The next extreme swing, when $t = 3T/4$, is negative, and numerically $= a_2 = ae^{-3\lambda T/4}$. The next is $a_3 = ae^{-5\lambda T/4}$; and so on. Hence $a_1/a_2 = e^{\lambda T/2}$.

The equations $q = Qa$, where $Q/C = 2\pi/T$, hold only for the amplitude $a$ which would exist if there were no friction. If there be a logarithmic decrement, $a_1$ and $a_2$ may be observed, and $a$ found from the formula

$$a/a_1 = \sqrt{a_1/a_2}.$$

**Ex. 1.**—The galvanometer of Ex. 2, 3 of § 87 had successive swings 15 and 13 and period 2 sec.; what was its logarithmic decrement?

Approximately, 2 is the diminution in 1 sec., *i.e.* a half period; if therefore $x$ = swing = 15, then $\dot{x} = -2$. But $\dot{x} = -\lambda x$, therefore $\lambda = 2/15$. More accurately, $\dot{x} = -2$ corresponds to the average value $x = \frac{1}{2}(15 + 13) = 14$, hence

$$2 = 14\lambda, \; \lambda = 1/7 = \cdot1428.$$

Strictly, in 1 sec. the logarithm to base 10 diminishes by log 15 − log 13 = ·06215 (to 5 places). Divide by ·43429 to reduce to Napierian logarithms; hence $\lambda = 6215/43429 = \cdot1431$. The approximate method is as good, in practice, as the more accurate.

**Ex. 2.**—The period is 1·6 sec., and successive swings are 11 and 10·4; find logarithmic decrement.

The half period is ·8, the charge in ·8 sec. is 11 − 10·4 = ·6, therefore we can put $-\dot{x} = -dx/dt = 6/8$. Also $x = \frac{1}{2}(11 + 10{\cdot}4) = 10{\cdot}7$; therefore

$$\lambda = -\dot{x}/x = 6/8 \times 10{\cdot}7 = \cdot0701.$$

**Ex. 3.**—If a quantity diminishing under viscous resistance halves its value in every $p$ seconds, and divides its value by 3 in every $q$ seconds, find the relations between $p$ and $\lambda$, $q$ and $\lambda$.

---

* This is an approximate statement, which involves $\lambda$ being a small quantity.

Taking Napierian logs, the logarithm diminishes by log 2 in time $p$, therefore $\lambda = \log 2/p$. In the same way $\lambda = \log 3/q$. It is a sufficiently close approximation to take $\lambda = \cdot 7/p = 1\cdot 1/q$ ; but the correct values of $\log_e 2$, $\log_e 3$ are ·69315, 1·09861.

**Ex. 4.**—If the first and second swings be 13 and 11 (in opposite directions) what is the corrected amplitude?

$$\alpha/13 = \sqrt{13/11},$$

or

$$\log \alpha = (3 \log 13 - \log 11)/2.$$

This gives $\alpha = 14\cdot 13$. But an approximate method is used when the logarithmic decrement is so small as this. The amplitude diminishes 13 − 11 or 2 in a half period. It is assumed it diminished half as much, or 1, in the previous quarter period. Hence

$$\alpha = 13 + 1 = 14.$$

The error here is only one per cent.

**Ex. 5.**—The first swing is 10·5 to right, the second is 9 to left, and the complete period is 2 seconds. Calculate the logarithmic decrement, and find after how long the swing will be less than 3 cm. Find also the corrected amplitude.

**Ex. 6.**—If such a viscous resistance were applied to the galvanometer of Ex. 5 that the second swing was a third of the first, find how much the apparent period would be affected.

**188. Condensers.**—Large capacities are necessary for Ballistic work, hence there are many plates arranged as in the figure. In practice they are sheets of tinfoil separated by mica or other thin dielectric. The practical unit of capacity is the Farad, it is the capacity of a condenser which holds a charge of one coulomb when the P.D. of its terminals is one volt. That is, the charge is $10^{-1}$ absolute units when the potential is $10^{8}$ absolute units, therefore the farad is $10^{-9}$ absolute units of capacity.

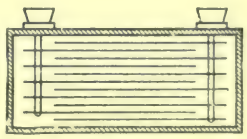

Fig. 93.

A coulomb is $3 \times 10^{9}$ electrostatic units of charge, and a volt is 1/300 electrostatic units of P.D., hence a farad is $9 \times 10^{11}$ units (electrostatic) of capacity. This is too

large for convenient manufacture, consequently standard condensers have their capacity given in microfarads or millionths of a farad. A microfarad is $9 \times 10^5$ absolute electrostatic units. If $q$, $V$, $K$, be the charge, potential and capacity in coulombs, volts and farads (or in absolute units) then $q = KV$.

Fig. 94.

Condenser keys are intended to charge and discharge condensers. A simple type is as shown, the insulating parts being shaded. The movable brass $DQ$ is connected to terminal $A$. The small spring presses it, when undisturbed, on the pad connected to terminal $B$, and pressure on the vulcanite at $P$ can disconnect it from $B$ and connect it rapidly to terminal $C$. In more complicated keys there is an arrangement for holding the key suspended for any arbitrary time in contact with neither $B$ nor $C$.

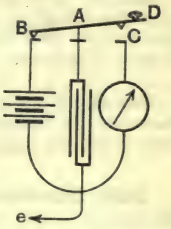

Fig. 95.

Let one pole of the condenser, battery, and galvanometer be connected to $A$, $B$, $C$ respectively, as in figure. Let the other poles be connected to earth. When the key touches $B$ the condenser is being charged by the battery, and when the key touches $C$ the condenser discharges through the galvanometer. If none of the conductors has too large a resistance we can assume the charging and discharging to be instantaneous. If therefore $V$ be the E.M.F. of the battery, the charge sent through galvanometer is $KV$. No resistances matter, unless they are very large (see § 189).

**Ex. 1.**—A 50 volt battery is used to charge a 1/3 M.F. (one third microfarad) condenser; which is then discharged through a galvanometer. The jump is 15 divisions. Find the ballistic constant ($q$).

Here
$$Q = 15q,$$
and
$$Q = KV = \tfrac{1}{3} \times 50 \times 10^{-6};$$
hence
$$q = 1{\cdot}11 \times 10^{-6}.$$

**Ex. 2.**—A Clark cell charges a condenser of 1 M.F. ; and the discharge through galvanometer produces a jump 3·4 cm. Find the ballistic constant.

$$3{\cdot}4q = KV = 10^{-6} \times 1{\cdot}43\,;$$

hence

$$q = 4{\cdot}2 \times 10^{-7}.$$

**Ex. 3.**—Would the working of the above (Ex. 2) be modified if the second swing (the reverse way) were observed to be only 3·0 cm. ?

Here (cf. Ex. 4, § 187) the amplitude diminishes by $3{\cdot}4 - 3{\cdot}0 = {\cdot}4$ in a half period. In a quarter period, the diminution should be ·2. Hence if the friction had been destroyed, the first swing would have been 3·6 nearly ; and

$$3{\cdot}6q = 10^{-6} \times 1{\cdot}43,$$

giving

$$q = 4{\cdot}0 \times 10^{-7}.$$

**Ex. 4.**—With the same condenser of unknown capacity we get kicks of 3·4, 9·2 when a Clark cell and an accumulator are used. What is the E.M.F. of the accumulator ?

Evidently the jumps are proportional to the E.M.F.'s, hence if the Clark be 1·43 volts the other cell is

$$1{\cdot}43 \times 9{\cdot}2 \div 3{\cdot}4 = 3{\cdot}87 \text{ volts.}$$

**Ex. 5.**—With the same battery, and with a standard M.F. condenser and another condenser, jumps of 15·5 and 9·3 are obtained. What is the capacity of the other condenser ?

$$93/155 = {\cdot}6 \text{ M.F.}$$

In experiments such as 4 and 5 it is never necessary to make a correction for frictional diminution of swing. For such a correction does not affect the ratio of the swings.

**Ex. 6.**—The current indicated per unit deflection of a galvanometer is $10^{-6}$ amperes. The period of swing is 2 sec. A condenser is charged by 5 Clark cells in series, and discharged through the galvanometer, producing a kick of 12 divisions. Find the capacity of the condenser.

**189. Discharge through Large Resistance.**—Let $K$ be the capacity of a condenser and $R$ the resistance through which it is being discharged. At time $t$, let the charge and potential be $Q$ and $V$, so that $Q = KV$. The difference of potential at the ends of the wire is $V$, hence the current is $V/R$. But the current equals the rate at

which $Q$ is decreasing, hence it $= -\dot{Q}$. We therefore have

$$\dot{Q} = -V/R = -Q/KR.$$

This is an equation of the form discussed in § 182, and its solution is $Q = Q_o e^{-t/KR}$, and therefore $V = V_o e^{-t/KR}$. The Napierian logarithms of both the charge $Q$ and the potential $V$ diminish at the rate $(KR)^{-1}$.

Again, if the condenser be being charged by a battery of E.M.F. $= E$, through a resistance $R$, and if $Q$, $V$, have above meanings, the current is $\dot{Q}$ and it $= (E - V)/R$. Hence

$$K\dot{V} = (E - V)/R.$$

If

$$x = E - V, \text{ this gives } \dot{x} = -x/RK,$$

$$\therefore\ x = x_o e^{-t/RK},$$

$x_o$ being the initial value of $x$. Choosing $t = o$ at the commencement of the charging, when $V = o$ and $x = E$, we have

$$E - V = Ee^{-t/RK},\ V = E\{1 - e^{-t/RK}\},$$

and

$$Q = KE\{1 - e^{-t/RK}\}.$$

The average current for time $t$ is

$$C = Q/t = \frac{KE}{t}\left\{1 - e^{-t/KR}\right\}.$$

If $R$ be not large, so that $t/KR$ is fairly large, $C = KE/t$. But if $t/KR$ is small enough for its square to be neglected, using the exponential theorem we have

$$C = \frac{KE}{t}\left\{1 - 1 + \frac{t}{KR}\right\} = E/R;$$

so that the current is the same as if the condenser had been absent. In ordinary telegraphy let $R$ be resistance of the live wire. Only one end of it is connected to the battery; the other is to earth, which can be regarded as a condenser so capacious that $t/KR$ is small. Consequently

the current is $C/R$, which implies that the effective resistance from one office to another is only the resistance of the wire one way.

**Ex. 1.**—A microfarad condenser discharges through a megohm. In how long a time is its charge (1) halved, (2) divided by 1000?

Here $K = 10^{-6}$, $R = 10^6$, therefore $KR = 1$ and $Q = Q_0 e^{-t}$. The charge is halved when $e^{-t} = 1/2$, $e^t = 2$, $t = \log_e 2 = \cdot 7$ sec. nearly. It is divided by 1000 when $t = \log_e 1000 = 3 \log_e 10 = 3 \times 2{\cdot}3 = 6{\cdot}9$ sec.

**Ex. 2.**—What would be the corresponding times if the same condenser were used as in Ex. 1, but the resistance were only 10000 ohms?

$KR = 1/100$, so the times are given by $100t = \cdot 7$, and $100t = 6{\cdot}9$. We infer that the discharge is practically instantaneous even when very large resistances are used. The methods of § 188 are valid with any resistances up to considerably over 10000 ohms; for most of the discharge takes place in a time very small compared to the galvanometer period.

**Ex. 3.**—If a condenser of 1/3 M.F. be discharged instantly through a galvanometer, the jump is 18. If the key remain suspended so that 2 sec. intervene between disconnecting the condenser from the battery and discharging it, the jump is only 7. Find the leakage resistance of the condenser.

During the two seconds the condenser is discharging through the enormous resistance, $R$ ohms, of its dielectric plates. The logarithmic decrement

$$= (\log_e 18 - \log_e 7)/2$$
$$= 2{\cdot}3\{\log_{10} 18 - \log_{10} 7\}/2 = \cdot 472.$$

Hence

$$KR = 1 \div \cdot 472;$$

but

$$K = 1/3 \times 10^6,$$

hence

$$R = 3 \times 10^6/\cdot 472 = 6{\cdot}35 \times 10^6 = 6{\cdot}35 \text{ megohms.}$$

**Ex. 4.**—The dielectric plates of a condenser are made of a material whose specific inductive capacity (in electrostatic units) is $k$, and whose specific resistance (in ohms per cubic cm.) is $r$. Given that an electrostatic unit charge is $1{\cdot}3 \times 10$ coulombs, and unit potential is 300 volts, show that the charge and potential of the condenser have a logarithmic decrement of $kr \cdot 12 \times 10^{11}$.

**Ex. 5.**—The specific resistance of mica is taken to be $9 \times 10^{15}$, and its dielectric constant or specific inductive capacity is 6·5. Apply the result of Ex. 4 to show that the logarithmic decrement is $4{\cdot}9 \times 10^{-5}$; and show that the charge will halve in about 4 hr.

**190. Comparison of Condensers. Sharing Charges.** Let $K_1$ and $K_2$ be the capacities, and let the insulation be perfect. First, if the capacities be small, as with condensers used in electrostatics. Connect $K_1$ to an electrometer and charge it to a potential $V$ giving deflection $x$. Of course one coat of the condenser and one terminal of the electrometer is earthed, and the other coat is connected to the other terminal. The second condenser (previously discharged) is now connected to the first. The charge $VK_1$ is shared and the total capacity is $K_1 + K_2$, hence the potential falls to $K_1V/(K_1 + K_2)$ and the deflection becomes $y$. Assuming that deflections are proportional to potentials we infer

$$y/x = K_1/(K_1 + K_2)$$

or

$$K_1/K_2 = y/(x - y).$$

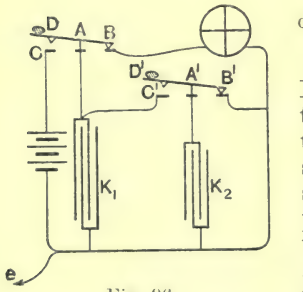


Fig. 96.

If $K_1/K_2$ is enormous, it is better, after sharing the charge, to disconnect $K_2$, discharge it, share again, and so on till the sharing operation has been repeated $n$ times. Each time, the potential is multiplied by $K_1/(K_1 + K_2)$. Hence if $z$ be the deflection after $n$ sharings, $z/x = \{K_1/(K_1 + K_2)\}^n$.

In Fig. 96, $ABC$ and $A'B'C'$ are condenser keys of the usual type (as in Fig. 94). If $D$ be pressed and then released, $K_1$ is first connected to the battery and then to the electrometer. In figure, $K_2$ is connected to earth by $C'$, and therefore is discharged. Pressing and releasing $D'$ makes it first share the charge of $K_1$ and then discharge itself; and this can be repeated many times with great rapidity.

Second, if $K_1$ be large enough we use a ballistic galvanometer instead of the electrometer. It is put in the same place, and the only difference is that the key $ABC$ should be capable of resting suspended in contact neither at $A$ nor $C$. First press $D$ and let the key fly back to $B$; thus

$K_1$ is charged by the battery and discharged through the galvanometer, the jump $x$ measuring $K_1V$. Next press $D$ to recharge $K_1$ and suspend the key; $K_1$ is now insulated and may be made to share its charge $n$ times with $K_2$ by working the second key. Finally the suspension is released and contact takes place at $B$; and the charge $V\{K_1/(K_1 + K_2)\}^n$ is sent through the galvanometer and measured by the jump $y$. Thus

$$x/y = \{(K_1 + K_2)/K_1\}^n.$$

**Ex. 1.**—A condenser is charged by a given battery and discharged through a galvanometer, giving a jump of 30 graduations. It is recharged, its charge is shared with a standard 1/3 M.F. condenser, and it is then discharged giving a jump of 18 cm. Find its capacity, and state the jump after two sharings.

$$K/(K + 1/3) = 18/30; \quad K = 1/2 \text{ in microfarads.}$$

After two sharings the jump is

$$30\{18/30\}^2 = 10{\cdot}8.$$

**Ex. 2.**—A standard 1/3 M.F. condenser shares its charge 20 times with a small condenser $K$, and the jump is thereby halved. Find $K$.

$$1/2 = \{1/3/(1/3 + K)\}^{20},$$
$$1 + 3K = 2^{1/20}, \quad \log(1 + 3K) = \tfrac{1}{20}\log 2.$$

As an approximation, the Napierian logarithm of $1 + 3K$ is $3K$ if $K$ is small. Hence

$$3K = \tfrac{1}{20}\log 2,$$
$$= {\cdot}693/20 = {\cdot}03465;$$
$$K = {\cdot}01155 \text{ M.F.}$$

Using common logs,

$$\log(1 + 3K) = {\cdot}3010/20 = {\cdot}01505 = \log 1{\cdot}0352;$$
$$3K = {\cdot}0352, \quad K = {\cdot}0117 \text{ M.F.}$$

**Ex. 3.**—If the capacities be $K_1$ and $K_2$, and the kick is halved in $m$ sharings ($m$ large), prove $K_2/K_1 = \cdot7/m$. If it be divided by three in $n$ sharings, prove

$$K_2/K_1 = 1{\cdot}1/n.$$

Obviously

$$\{(K_1 + K_2)/K_1\}^m = 2, \quad (K_1 + K_2/K)^n = 3.$$

Taking Napierian logarithms, and neglecting squares of small quantities,

$$\log (1 + K_2/K_1) = K_2/K_1 = \frac{1}{m} \log 2 = \cdot 7/m,$$

$$K_2/K_1 = \frac{1}{n} \log 2 = 1 \cdot 1/n.$$

More accurate values are $\cdot 693/m$, $1 \cdot 098/n$.

**Ex. 4.**—If the charge of $K_1$ be shared with $K_2$, and if $K_1$ also leak for time $t$, show that the final charge is the same whether the sharing occurs before, after, or during the leaking.

If before or after, the sharing multiplies the charge by $K_1/(K_1 + K_2)$, and the leakage multiplies by $e^{-\lambda t}$; and the order of multiplication is indifferent. If during, then a time $t_1$ comes before, and $t - t_1$ after, the sharing. If the time $t_1$ be taken after sharing, the charge at time $t_1$ is unaltered and therefore the same leakage takes place subsequently. Hence the final effect is the same. It follows that leakage and sharing can be separately calculated.

**Ex. 5.**—A large condenser $K$ is shared with a small one $k$ a hundred times, and the jump is multiplied by $\cdot 3$. The time taken is noted, and when $K$ is recharged and allowed to leak for that time the jump is multiplied by $\cdot 9$. Find $k/K$.

The diminution due to sharing only is $\cdot 3/\cdot 9 = 1/3$; hence

$$100\, k/K = 1 \cdot 1 \text{ (see Ex. 3)};$$

$$\text{therefore} \qquad k/K = \cdot 011 = 1/90 \text{ nearly.}$$

**Ex. 6.**—A standard condenser is constructed thus. It consists of 50 plates of which the effective area of each is 100 sq. cm.; the dielectric is air, and its thickness is 1/2 mm. Find its electrostatic capacity. Show that it would have to be connected (as above) about 80 times to a charged M.F. condenser to halve its potential.

By § 55 the capacity $= A/4\pi d$. The total area $A = 50 \times 100$; and $d = 1/20$; hence the capacity $k = 20 \times 50 \times 100/4\pi = 8000$ nearly, 7958 more accurately. If $K$ be a microfarad, its capacity in electrostatic units is $9 \times 10^5$. Hence

$$k/K = 8 \cdot 842 \times 10^{-3}.$$

The number of connections

$$n = \cdot 693 \div (k/K) = 78.$$

Experiments of this kind can be used to compare the electrostatic and the electromagnetic units of capacity. It is obviously difficult, however, to determine the precise contact of the key which exactly halves the potential; and it is also difficult to

standardise electrostatically and electromagnetically the two condensers used.

**Ex. 7.**—An electrostatic standard condenser is prepared whose capacity is 1000. A standard microfarad is charged, and made to share its charge 500 times with the smaller condenser. In what ratio is its charge diminished?

**Ex. 8.**—Two condensers (which do not leak) are placed in parallel and charged, and are then separately discharged through a ballistic galvanometer. Show the kicks are proportional to their capacities. What difference would it have made if they had been charged in series?

**Ex. 9.**—A Leyden jar, capacity 500 electrostatic units, is charged and is allowed to discharge itself slowly by conduction through a very bad conductor (*e.g.* a damp silk thread). It is observed that its charge is halved in ten minutes. Calculate the resistance of the conductor in megohms.

**191. Rotating Commutators.**—In Fig. 97, the circular wheel $A$ has its insulating parts shaded. The conductors may all be connected to a condenser $K$ whose other pole is connected to spring $B$, while springs $B$, $C$ are connected to poles of a battery or otherwise kept at a P.D. equal to $V$. As the wheel rotates, the condenser is charged to potential $V$ whenever the metal of the wheel touches $C$, and is discharged whenever the metal touches $B$. We shall suppose these operations instantaneous. The charge given at each contact is $KV$. If there be $n$ contacts at either spring per second, the total charge transferred per second from $C$ to $B$ is $nKV$, and if a galvanometer be put in series with $CB$ at either of the positions $G_1$ or $G_2$ in figure it will indicate an average current $nKV$, provided of course that the period of the galvanometer needle is considerably larger than $1/n$. Thus the condenser and commutator act precisely as a resistance of magnitude $R' = 1/nK$ would act with the same constant P.D. This

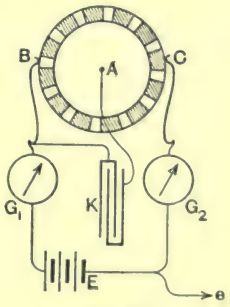

Fig. 97.

fictitious resistance does not obey the same laws as ordinary resistances if put in series or parallel with other resistances. Its effect is always calculable by higher mathematics, but the only really simple case is that in which we can assume a constant potential difference and an instantaneous charge and discharge.

**192. Comparison of Capacities. Zero Methods.**—The Method of Mixture is shown in Fig. 98. Let $c$ be the current which flows through $R_1$ and $R_2$ at make. The condensers are charged to P.D.'s $= cR_1$, $cR_2$ and the charges that run into them through the galvanometer collectively

$$= -cK_1R_1 + cK_2R_2.$$

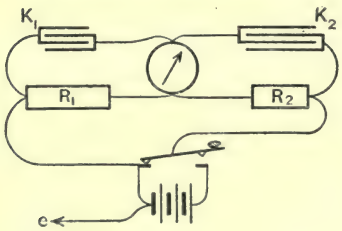

Fig. 98.

The same charge flows back at break. The galvanometer is unaffected if the charge is zero, or $K_1R_1 = K_2R_2$. The condenser charges are considered to mix on their way, whence the name.

In the second method the connections are as in Fig. 99. When a balance is reached, nothing goes through the galvanometer at make or break. At make, initially, both ends of the galvanometer have been connected to the same pole of the battery, and therefore are at the same potential. The potentials at any subsequent short time $t$ will also be the same (§ 189) if

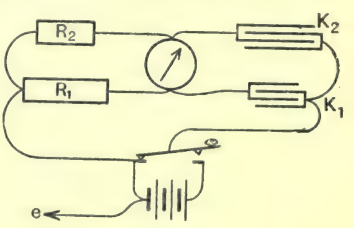

Fig. 99.

$$V\left\{1 - e^{-t/K_1R_1}\right\} = V\left\{1 - e^{-t/K_2R_2}\right\}.$$

Hence a balance is reached as before if $K_1R_1 = K_2R_2$.

In both these methods the galvanometer is used ballistically. But if a rotating commutator be used we have a steady current and the condensers behave precisely like resistances $1/nK_1$, $1/nK_2$. Both methods of connecting are then seen to be cases of the Wheatstone Net.

It may be noted that these methods of comparing resistances are much less affected by condenser leakage than the method of § 190.

## GENERAL EXAMPLES.

**Ex. 1.**—Express the capacity of the earth, regarded as an isolated sphere, in farads.

Its radius $= 4 \times 10^9/2\pi$ cm., hence this is its capacity in electrostatic units. A farad is $9 \times 10^{11}$. Hence if $E$ be the capacity of the earth in farads, $E = 2/900\pi = \cdot 707 \times 10^{-3}$. Hence evidently $E = 707$ M.F. ; and discharging a M.F. condenser to earth is not an absolute discharge, but a sharing the charge with a much larger capacity.

**Ex. 2.**—If a toothed wheel of 20 teeth were connected as a rotating commutator to a M.F. condenser, how fast would it have to rotate in order to make the equivalent resistance = 10000 ohms?

$R = 1/nK$ ; but $R = 10^4$, $K = - 10^6$, $\therefore n = 100$. There must be 100 contacts per sec., therefore the wheel turns 100/20 or five times per sec.

**Ex. 3.**—The condenser of Question 6, § 190, is compared with a standard M.F. by a zero method. If 1 ohm be the resistance connected immediately to the M.F., find the resistance $R$ connected to the other condenser.

$$R = 9 \times 10^5/7958 \text{ (nearly)} = 113.$$

**Ex. 4.**—A gold leaf electroscope of capacity 6 electrostatic units leaks so fast that its charge is halved in 10 minutes. Find its insulation resistance.

In farads, $K = 6/9 \times 10^{11}$. The logarithmic decrement $= \log_e 2 \div 600$ and also $= 1/KR$. Hence

$$6R/9 \times 10^{11} = 600/\log_e 2 = 600 \cdot 7,$$

$$\therefore R = 9 \times 10^{13}/\cdot 7 = 13 \times 10^{13} \text{ ohms}$$

$$= 13 \times 10^7 \text{ megohms.}$$

**Ex. 5.**—If the above electroscope (Ex. 4) be charged to a potential of 200 volts, find the current.

$$\text{Current} = 200/13 \times 10^{13} = 1{\cdot}54 \times 10^{-12} \text{ amperes.}$$

Dividing by $3 \times 10^9$ we get $5{\cdot}1 \times 10^{-19}$ electrostatic units.

These Examples, 4 and 5, show how the largest resistances and the smallest currents are measured.

**193. Condenser and Resistance in Parallel.**—A condenser of capacity $K$ is placed in parallel with a resistance $R$ in any circuit containing a make and break key. If the final current in the wire at make is $C$, the P.D. between the ends of the wire $R$ is $RC$ and hence the charge in the condenser is $KRC$. Hence during make, while the current rises from zero to its full value, a little more electricity flows towards the + end of $R$, and from the − end of $R$, than flows through $R$ itself. The excess is $KRC$, and goes into the condenser. If we could arrange an apparatus to measure the difference of total flow between the parts $AB$ and $BC$, or $CD$ and $BC$, of the circuit, it would register the excess charge $KRC$.

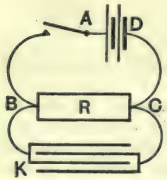

Fig. 100.

# CHAPTER XI.

## INDUCED CURRENTS.

**194. Mutual Energy.**—Consider two permanent magnets or other magnetic systems exerting forces one on the other. Their Mutual Energy (or Mutual Potential), M.E., is the work that will be done by the magnetic forces if either system be removed to an infinite distance from the other, each system remaining unchanged in itself. In this chapter we shall consider the systems to be surrounded by air or other media which are magnetically equivalent to air, and the next chapter will be devoted to the special effects of different media.

**195. Formulae for Mutual Energy.**—If one of the two systems be a unit magnetic $N$ pole at a point $O$, the M.E. is the potential at $O$ due to the other system (see Def. § 140). If this other system be a current of strength $c$ absolute units, flowing in any circuit, subtending a solid angle $\Omega$ at $O$, the potential, and therefore the M.E., is $c\Omega$. If the pole be of strength $m$, the M.E. becomes $cm\Omega$. But $4\pi m$ is the total number of lines of force that radiate in every direction from a pole $m$, hence $m\Omega$ is the number of the lines due to $m$ that fall within a solid angle $\Omega$, *i.e.* that pass through the circuit of $c$. Hence the M.E. is $=$ $c \times$ number of lines of force through $c$ due to $m$.

Further, if the system be any magnet or system of magnets, it can be supposed replaced by a series of poles $m$ situated at points $O$. The M.E. is then the algebraic sum of the numbers of the lines of force, due to each separate pole, which pass through the circuit. But the sum of the numbers of the separate lines from different point-poles which go through any circuit is (as in § 20) the same as

the number of the resultant lines due to the aggregate of the point-poles, which go through the same boundary. A current system can be regarded as a system of magnetic shells.

Hence, the M.E. of two systems, one of which is any whatever and the other is a current $c$ (absolute units), is the product of $c$ by the number of lines of force due to the first system which traverse the circuit of $c$. The number of lines through a circuit may be called the Flux through that circuit, and will be denoted by $F$. Hence Mutual Energy $= W = Fc$.

**196. Change of Flux.**—If the first system be either moved or altered, $F$ changes its value by an amount $\delta F$, and the rate of change is $\dot{F}$ or $dF/dT$ (§ 180). If $c$ does not alter, then

$$W = Fc \text{ gives } \delta W = c\delta F,$$

and therefore

$$\dot{W} = c\dot{F}.$$

Now $\dot{W}$ = the rate at which the mutual energy is being increased. The increase must be caused by work provided in causing the motion or alteration of $F$.

But (§ 102) E.M.F. is the work supplied (reversibly) per unit current per second. Evidently $\dot{W}$ is reversible, for it changes sign if $c$ does; hence the E.M.F. due to the motion is $E = \dot{W}/c = \dot{F}$. The law that Induced E.M.F. = Rate of Change of Flux is known as Faraday's Law. It can be written $E = \dot{F}$. Of course $E$ is in absolute units. To reduce to volts we divide by $10^8$ (§ 103).

The law can be differently expressed. $F$ = the number of lines enclosed by the circuit $c$, so $\delta F$ = the number of lines cut by the circuit in the change considered, and $\dot{F}$ = number of lines cut per sec.

**197. Lenz's Law.**—This gives the direction of the E.M.F. produced. The Induced E.M.F. is so directed that the magnetic field of the resulting current opposes the change of flux.

For the mechanical forces acting between two systems tend to produce a motion which provides mechanical work at the expense of their mutual energy; and therefore diminishes $W$. Consequently, the spontaneous motion, if permitted, makes $\dot{W}$ negative, and therefore $\dot{F}$ negative; and so tends to diminish $c$. Hence $\delta c$ is of contrary sign to $c$, and would produce motion of contrary sign to the spontaneous motion; hence the change of flux induces a current which opposes the change. If $\dot{W}$ be reversed, so is $\delta c$, therefore the law holds in this case also.

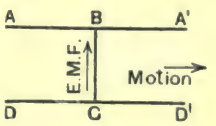

Fig. 101.

**Ex. 1.**—A train runs in London with a velocity of 40 miles an hour, over insulated rails whose distance apart is $1\frac{1}{2}$ metres. Find the P.D. thereby produced between the rails.

We take 11 yards = 10 metres, and therefore a mile = 1600 metres = $1{\cdot}6 \times 10^5$ cm. In 1 sec. the train moves $40 \times 1{\cdot}6 \times 10^5/3600$ cm., and it sweeps out an area $40 \times 1{\cdot}6 \times 10^5 \times 150/3600$ sq. cm. The earth's vertical field is ·438, there are therefore ·438 lines to every horizontal square cm., so the number of lines cut per sec.

$$= \dot{F} = 40 \times 1{\cdot}6 \times 10^5 \times 150 \times {\cdot}438/3600$$
$$= 1{\cdot}17 \times 10^5.$$

It follows that the E.M.F. in the wire is $1{\cdot}17 \times 10^5$ absolute units. In volts it is $1{\cdot}17 \times 10^{-3} = {\cdot}00117$.

If $BC$ in figure is the train, the downward flux is increasing in the area $ABCD$ to left of $BC$. A current which would oppose this increase would produce an upward field in the area $ABC'D$. Similar reasoning shows that it would produce a downward field in the area $A'BCD'$ in which the downward flux is diminishing. Hence the current, if it existed, would flow from $B$ to $C'$; and therefore the E.M.F. is in the direction $BC'$. No current can endure between insulated rails, and therefore the algebraic sum of the E.M.F. in $BC$ and the P.D. of $B$ over $C$ is zero. Hence, when the steady state is reached, $B$ is at the higher potential. The rail $ABA'$ is higher than $DCD'$ by ·00117 volt. The rail of higher potential is on the left hand side of the travelling train. Of course the result would be contrary if the vertical component of the earth's field were upwards.

**Ex. 2.**—A plane frame of any shape, and area $A$, has $N$ turns of wire wound around it. The frame is rotated with angular velocity $\omega$ about a vertical axis in (or parallel to) its plane. The two ends

of the wire are soldered to brass terminals $L$ and $M$ which rotate with the frame and make contact with two springs $X$ and $Y$ for only a very short part of each rotation, while the frame is passing through coincidence with the magnetic meridian. Find the P.D. produced between the springs.

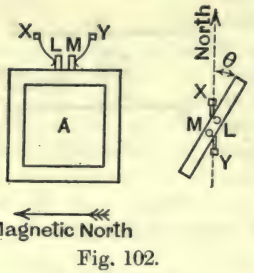

Fig. 102.

When $\theta$ is the angle rotated through, the component perpendicular to the plane $A$ of the earth's magnetic field $= H \sin \theta$, $= H\theta$ in limit if $\theta$ be small. The number of lines of force through $A$ is $HA\theta$; but there are $N$ turns, and therefore the number of lines through the circuit $= F = NHA\theta$. Hence $\dot{F} = NHA\dot{\theta}$. But the rate of change of $\theta$ is the angular velocity, or $\dot{\theta} = \omega$. Hence the induced E.M.F. is $NHA\omega$, so long as the frame is almost in the meridian. But it never touches the springs $X$ and $Y$ except at such times, therefore $NHA\omega$ is the P.D. produced in the springs.

**Ex. 3.**—$A$ is a square frame of 50 cm. side. There are 1000 turns. $X$ and $Y$ are connected in series with a Clark cell and a galvanometer. It is evident that no current can ever pass unless $X$ and $Y$ are touching the terminals of the wire; and that no current will pass even then if the E.M.F. due to the motion is equal and opposite to that of the Clark. Find how many turns the frame must make per sec. for an exact balance.

The induced E.M.F. is $NHA\omega$. The E.M.F. of a Clark is 1·43 volts $= 1{\cdot}43 \times 10^8$ absolute units. Hence

$$1{\cdot}43 \times 10^8 = NHA\omega = 1000 \times {\cdot}186 \times 2500 \times \omega,$$

and the number of turns per sec.

$$= \frac{\omega}{2\pi} = 49 \text{ nearly.}$$

**Ex. 4.**—The same frame as in Ex. 3 is rotated 10 times a second. The apparatus is connected in series with a galvanometer and put in a potentiometer circuit. A balance is obtained at a distance 45 cm. of potentiometer wire, while a Clark cell balances at 220 cm. Find the E.M.F. of the Clark.

The E.M.F. of the frame, in volts,

$$= 1000 \times {\cdot}186 \times 2500 \times \omega/10^8$$
$$= 1000 \times {\cdot}186 \times 2500 \times 20\pi/10^8 = \pi \times {\cdot}186/2 = {\cdot}2922.$$

The E.M.F. of Clark

$= \cdot 2922 \times 220/45 = 1 \cdot 428 = 1 \cdot 43$ nearly.

It is evident that Examples 3 and 4 indicate methods by which any constant cell can be standardised in absolute units.

**Ex. 5.**—Find the maximum E.M.F. produced by rotating a circular coil of 500 turns, 20 cm. radius, 10 times per second.

**198. Impulsive E.M.F.** or **Electromotive Impulse.**—If $\delta q$ be the charge which goes round a circuit in the interval $\delta t$, $\delta q/\delta t$ is the average current. In the limit (§ 180) $dq/dt = c$.

If $c$ be a current entirely due to induction, and $R$ be the resistance of the circuit in which it flows, we have

$$c = E/R,\ E = dF/dt,$$

$$\therefore \frac{dq}{dt} = \frac{1}{R}\frac{dF}{dt}.$$

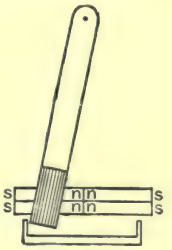

Fig. 103.

It follows that $q = F/R +$ a constant. If $F$ be the final flux and $F_0$ the initial flux, the constant is determined by the condition that no current is induced, and $q = 0$, if $F$ remains $= F_0$; hence

$$q = F/R - F_0/R = (F - F_0)/R = \delta F/R.$$

Of course $\delta F$ simply signifies the change in $F$, and there is no suggestion that this change is small. The charge $q$ is measurable by a ballistic galvanometer.

The quantity $\delta F$ is called the Impulsive E.M.F. We shall also use the name Electromotive Impulse, and the abbreviation E.M.I. We therefore have

Electromotive Impulse = Charge ÷ Resistance.

To reduce $\delta F$ to practical units we divide (as in § 195) by $10^8$. The Practical unit is the impulse due to a volt enduring for a second.

**Ex. 1.**—A close coil of small radius (1 cm. *e.g.*) consisting of 20 turns of moderately coarse wire is connected in series with a galvanometer of 5 ohms resistance. The *N.* pole of a simple magnet of

pole strength 80 is thrust well through the coil. Calculate approximately the charge that goes through the galvanometer.

The number of lines originating from the pole is $4\pi . 80$. Making the approximate assumption that each of them is cut 20 times, once by each turn, we get $\delta F = 4\pi \times 80 \times 20$. If $U$ be the E.M.I. in volt-seconds, $U = 4\pi \times 80 \times 20 \times 10^{-8}$. The total resistance of the circuit can be taken as 5, the resistance of the galvanometer; for a few turns of coarse wire can be neglected. Hence

$$Q = 4\pi \times 80 \times 20 \times 10^{-8}/5$$
$$= 4 \times 10^{-6} \text{ coulombs.}$$

**Ex. 2.**—A set of permanent magnets is arranged as in Fig. 103, the aggregate pole strength of the $N$. poles in contact being 500. There is a movable coil of 120 turns which can be thrust rapidly from one position to the other, so as to pass completely over the $N$. poles. Its resistance is 2 ohms. Estimate its E.M.I., and the charge which it would send through ballistic galvanometers of resistances $\frac{1}{2}$ ohm, 5 ohms, 400 ohms.

Let $U$ be the E.M.I. in practical units, then roughly

$$U = 4\pi \times 500 \times 120/10^8 = 7{\cdot}54 \times 10^{-3}.$$

Of course all the lines are not cut, so $U$ is actually smaller than this. The total resistances in the three cases are $2\frac{1}{2}$, 7, 402, hence the charges are somewhat less than

$$3{\cdot}016 \times 10^{-3},\ 1{\cdot}077 \times 10^{-3},\ 1{\cdot}875 \times 10^{-5}.$$

An arrangement of magnets of this character can be used for ballistic work in the same way as a battery is used for current work. We shall call it a Magnet Inductor.

**Ex. 3.**—A Magnet Inductor of negligible resistance produces a kick of 25 divisions with a given galvanometer. If a 50-ohm coil be put in series, the kick falls to 19 divisions. Neglecting the corrections that ought to be applied for logarithmic decrement, find the resistance of the galvanometer. If the E.M.I. $= 7{\cdot}54 \times 10^{-3}$, find the ballistic constant (§ 185) of the galvanometer.

Let $G$ be the required resistance. The charges are $U/G$, $U/(G + 50)$. But their ratio is 25 to 19. Hence

$$25G = 19(G + 50)\ ;\ G = 19 \times 50/6 = 158 \text{ ohms.}$$

$U = 7{\cdot}54 \times 10^{-3}$, hence the charge denoted by 25 divisions is $U/G = 7{\cdot}54 \times 10^{-3}/158{\cdot}3$.

The galvanometer constant $q$ = charge denoted by one division,

$$= 2 \times 10^{-6}.$$

[If the resistance of the inductor be 2 ohms, the corrected galvanometer resistance is $158 - 2 = 156$, but $q$ is unaltered. The kind of correction to be applied to $q$ is indicated later. In experimental work the E.M.I. of a magnet inductor could not be calculated as in Ex. 2, but would be found experimentally.]

**Ex. 4.**—A flat circular coil (Delezenne Circle, or Earth Inductor) of 225 turns of wire of mean radius 18 cm. is held exactly horizontal. It is rapidly reversed so that its position is again horizontal. Taking the earth's vertical field as $V = \cdot 44$, find $U$ the E.M.I. due to reversal.

The area of one turn is $\pi . 18^2 = 324\pi$. The total area $A$ of all the turns $= 324\pi \times 225 = 72900\pi$. The flux is $AV$, and the change of flux due to complete reversal is $2AV$. Hence

$$\delta F = 2 \times 72900\pi \times \cdot 44 = 201500,$$
$$U = \delta F \times 10^{-8} = \cdot 0020.$$

A Delezenne circle can be used as the ballistic analogue of a constant battery, for its E.M.I. is calculable and unchanging.

**Ex. 5.**—Calculate the E.M.I. of the above coil if held vertically in an $E.$ and $W.$ plane (perpendicular to the magnetic meridian) and suddenly reversed. Given $H = \cdot 186$.

The calculation is the same except that the horizontal field $\cdot 186$ replaces the vertical one $\cdot 44$. The result is $\cdot 000850$.

**Ex. 6.**—If the above circle were reversed with respect to the earth's vertical field, and were in series with a galvanometer of (1) $\frac{1}{2}$ ohm, (2) 450 ohms, find the charges sent round. The resistance of the coil is given as 10 ohms. If the jumps in the two cases be 17 and 11, find the ballistic constants of the galvanometers.

*Note.*—This Example indicates a method of standardising a ballistic galvanometer. We have three principal methods: (1) Find the current constant and the vibration period, and deduce the ballistic constant as in § 185; (2) Notice the deflection due to the discharge of a standard condenser charged by a known battery, as in § 188, Ex. 1; (3) By the Earth Inductor. A fourth method is by the formula of § 185.

**Ex. 7.**—A small coil of 200 turns of 1·5 cm. mean radius is placed in the intense magnetic field between the pole pieces of a powerful electromagnet. The plane of the coil is perpendicular to the lines of force. The coil is in series with a galvanometer whose ballistic constant is $4 \times 10^{-7}$, and the resistance of the circuit is

450. When the coil is jerked right out of the magnetic field, the jump is 10 divisions. Find the field strength. Here

$$Q = 4 \times 10^{-7} \times 10 = 4 \times 10^{-6},$$

and

$$U = QR = 4 \times 10^{-6} \times 450 = 1{\cdot}8 \times 10^{-3}.$$

Hence

$$\delta F = 1{\cdot}8 \times 10^{5}.$$

Now the total area of the coil

$$= \pi \times 1{\cdot}5^2 \times 200 = 450\pi.$$

Hence the field = flux per unit area

$$= 1{\cdot}8 \times 10^5/450\pi = 400/\pi = 127.$$

The method indicated is quite suitable for measuring such fields. If the gap between the pole pieces is wide enough, the field could also be measured by suddenly reversing the coil, *i.e.* rotating it through 180°.

**Ex. 8.**—If the circle of Ex. 4 be rotated about a vertical axis 30 times per second, connected by springs as in Fig. 102, except that the contacts are continuous for practically the whole of each half turn, and are reversed as the circle passes through the E. and W. position, find the average E.M.F. produced.

By Ex. 5, an E.M.I. of ·00085 is produced in each half turn, and therefore 60 times per second. The average E.M.F. = ·00085 × 60 = ·051 volt.

**Ex. 9.**—Find the E.M.F. (average) if the coil be rotated as in Ex. 8 but about a horizontal axis, the connections being reversed every time the plane of the coil is horizontal.

**Ex. 10.**—A simple Dynamo consists of a coil of 5000 turns rotated 20 times per second in a magnetic field whose maximum flux through each turn is 5 absolute units. Find (1) Maximum instantaneous E.M.F., (2) Average E.M.F. per half turn under the most favourable conditions, (3) E.M.F. of mean square, *i.e.* that E.M.F. whose square = average value of the square of the E.M.F. produced.

**199. Consequences of Lenz's Law.**—In § 196 we saw that the induced current has a magnetic field which opposes the relative motion of the two systems.

The induced current $\propto \dot{F}$, and therefore $\propto$ the relative velocity of the systems, other things being equal. The

field of the current $\propto$ the current; and therefore the total force called into play is proportional to the velocity and opposes it. Consequently, it produces the same effect as viscous friction.

This effect is discussed generally in §§ 182, 183. It produces a logarithmic decrement, $\lambda$, proportional to the retardation per unit velocity. Given $F$, the current $\propto 1/R$, the conductivity of the conductor. Consequently $\lambda \propto 1/R$. This $\lambda$ is the logarithmic decrement due to induced currents. The viscosity of the air produces another logarithmic decrement.

If the magnet of a moving needle galvanometer swings within a metal ring, or the coil of a moving coil galvanometer be packed in a cylindrical metal box *ABCD* (Figures 104, 105), we have large transient currents induced. Oscillations are either impossible or are rapidly destroyed, and the galvanometer indicates almost immediately, with perfect steadiness, the final reading which indicates the current in the coil. Such a galvanometer is unsuitable for ballistic work, and is called "dead beat."

Fig. 104.

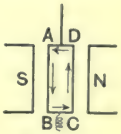

Fig. 105.

**Ex. 1.**—In an ordinary oscillation magnetometer there is no appreciable logarithmic decrement. If a flat plate of copper be laid under the magnet, the amplitude is observed to be halved in fifty swings. If a precisely similar plate of zinc be used, the amplitude is halved in 180 swings. Compare the specific resistances of the metals.

Whatever be the distribution of the currents, they will be the same in both cases. The magnitudes, however, are inversely proportional to the resistances of corresponding portions of the conductors. The plates are alike in all other respects and therefore the currents (for the same velocity of magnet) vary inversely as the specific resistances. The specific resistances are therefore proportional to the logarithmic decrements, and therefore inversely proportional to the time in which the amplitude is halved. Hence

Specific resistance of zinc : specific resistance of copper

$$= 180 : 50 \qquad = 3{\cdot}6 : 1.$$

**Ex. 2.**—In how many swings would the above amplitude be halved if both plates were used, one resting on the other?

If $N$ be the number of swings, then $N$, 50, 180 are in the ratio of resistances of corresponding portions of the compound plate and of the copper and zinc. But the resistance in the first case is that of the copper and zinc is parallel, therefore

$$1/N = 1/50 + 1/180\,;\ N = 39.$$

**Ex. 3.**—How thick a zinc plate would be as effective as the copper plate?

Obviously the zinc plate conducts as well as the copper plate if its thickness is 3·6 times as great.

**Ex. 4.**—A ballistic galvanomenter has a very small logarithmic decrement. When its terminals are connected by zero resistance, the amplitude halves in 30 seconds. When they are connected through a resistance of 50 ohms, the amplitude halves in 38 seconds. Find the resistance of galvanometer.

The logarithmic decrements in the two cases are ·69/30, and ·69/38. If $R$ be the galvanometer resistance, these are inversely proportional to $R$ and $R + 50$. Hence $R/(R + 50) = 30/38$, $R = 237{\cdot}5$.

**Ex. 5.**—A galvanometer of 450 ohms resistance has a logarithmic decrement, due to entirely mechanical causes such as the resistance of the air, which halves its swing after 30 swings. If a resistance of 5000 ohms be inserted, the swing is halved in 20 swings. How soon will the swing be halved if the galvanometer be simply short-circuited?

Let $N$ swings be the number required. The logarithmic decrements in the three cases are proportional to 1/30, 1/20, $1/N$. But logarithmic decrements due to different causes are simply additive (cf. § 190, Ex. 4), hence the parts of the logarithmic decrement due to induced currents are proportional to $1/20 - 1/30$, $1/N - 1/30$. These are in the inverse ratio of the total resistances. Hence

$$1/N - 1/30 : 1/20 - 1/30 = 5450 : 450$$

$$\frac{1}{N} - \frac{1}{30} = \frac{5450}{50 \times 60} = \frac{109}{540}$$

$$\frac{540}{N} = 18 + 109 = 127\,;\ N = 4{\cdot}3.$$

This example illustrates the difficulty of using ballistic galvanometers shunted. It also illustrates another point. If the swing is halved in 4·3 swings (*i.e.* 4·3 half-periods), the common logarithm of the swing diminishes in a quarter period by

$$\log 2/8{\cdot}6 = {\cdot}3010/8{\cdot}6 = {\cdot}0373 = \log 1{\cdot}09.$$

Consequently the first swing observed ought to be corrected by being multiplied by 1·09 (see § 187) when accurate ballistic measurements are taken.

**Ex. 6.**—A ballistic galvanometer, excited by condenser discharge, has period 2·50 sec. If excited by an earth inductor, its period rises to 2·56. Assuming the logarithmic decrement is zero in the first case, what is it in the second case?

With the notation of § 199,

$$2\cdot50 = 2\pi/n,\ 2\cdot56 = 2\pi/m, \text{ and } m^2 = n^2 - \lambda^2.$$

Hence

$$\lambda^2 = 4\pi^2\ \{(1/2\cdot50)^2 - 1/(2\cdot56)^2\},\ \lambda = \cdot54.$$

This method of observing $\lambda$ is usually a bad one, for a slight alteration of period is only detected in a large number of swings, and very few swings can be observed if $\lambda$ is at all large.

**Ex. 7.**—A suspended metal lamina is oscillating, in its own plane, in a strong magnetic field. Show that there will be a logarithmic decrement proportional to the field and inversely proportional to the resistivity of the lamina.

**Ex. 8.**—A circular copper disc is smoothly pivoted at its centre so that it can rotate in its plane about a vertical axis. If it be started rotating in a strong steady magnetic field, show that it will be viscously retarded. If it be in a magnetic field which is made to rotate about the vertical axis, show that it will ultimately rotate with the field.

**200. Mutual Induction.**—Let both of the systems whose M.E. is $W$ be currents in circuits. If $c_1$ be the current in the first or primary circuit, the field at every point in space due to this circuit is proportional to $c_1$. It must be remembered that this is only strictly true under the conditions assumed throughout this chapter, that there is no medium in the magnetic field whose magnetic properties are fundamentally different from those of air; in particular, no steel or soft iron.

As the field is proportional everywhere to $c_1$, the flux $F$ through the second or secondary circuit must also be proportional to $c_1$, therefore $F = Mc_1$, and $W = Fc$ (as in § 195), therefore $W = Mc_1c$ where $M$ is an absolute constant depending only on the dimensions and relative position of the two circuits, and $C$ is the current in the second one.

Since $F = Mc_1$, $F$ must $= M$ if $c_1 = 1$. The constant $M$ is called the Coefficient of Mutual Induction (or

simply the Inductance) and the symmetry of the relation $W = Mcc_1$ shows that its value is unaltered if the first and second circuits be interchanged. Hence the definition,—

The **Coefficient of Mutual Induction** of two circuits is the Total Magnetic Flux due to unit current in either circuit which passes through the other.

In this definition absolute units are implied. The Practical unit for this coefficient is $10^9$ Absolute units and is called the **Henry.**

**201. Induced E.M.F.**—If $c_1$ be altering, let $E$ be the E.M.F. induced by this alteration in the other circuit. In absolute units, $E = \dot{F} = M\dot{c}_1$, hence $M = E/\dot{c}_1$. If however $E$ and $c$ were in volts and amperes, these quantities in absolute units would be $10^8E$ and $10^{-1}c_1$, so that $M$ in absolute units $= 10^9E/c_1$. The multiplier $10^9$ is avoided by the use of the Henry in practice. We see that another definition of $M$ can be adopted,—

The **Coefficient of Mutual Induction** of two circuits is the E.M.F. induced in either when the current in the other is increasing at unit rate. The practical unit is the Henry and is the mutual inductance of two such coils that an E.M.F. of one volt is induced in either if the current in the other is increasing at the rate of one ampere per sec.

If $c_1$ receives a finite alteration $\delta c_1$, then $F$ always $= Mc_1$, and $M$ is constant, hence

$$\delta F = M\delta c_1.$$

But $\delta F$ is E.M.I. in the secondary, and $= qR$ (§ 198), hence

$$M\delta c_1 = qR.$$

We have therefore a third definition of $M$. The **Coefficient of Mutual Induction** is the impulsive E.M.F. produced in either by a unit change of current in the other. $M$ is in Henries if we reckon the other quantities in terms of the ampere, volt and second.

**Ex. 1.**—The primary circuit contains a battery, a tangent galvanometer or milliameter (T.G. in Fig. 106) and a tapping key for make and break. When the current is flowing, it measures 35 milli-amps. The secondary is connected to a ballistic galvanometer (B.G.) whose constant is $4 \times 10^{-7}$ coulombs. At make or break of primary, the jump is 7·3 divisions. Find the coefficient of mutual inductance, given that the resistance of secondary circuit is 550 ohms.

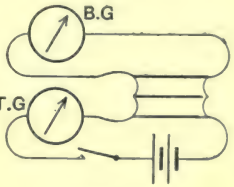

Fig. 106.

Here

$$q = 7{\cdot}3 \times 4 \times 10^{-7} = 2{\cdot}92 \times 10^{-6},$$
$$\delta c = c - o = 3{\cdot}5 \times 10^{-2},$$

and

$$M = Rq/\delta c$$
$$= 2{\cdot}92 \times 550 \times 10^{-6}/3{\cdot}5 \times 10^{-2}$$
$$= 4{\cdot}6 \times 10^{-2} \text{ Henries.}$$

**Ex. 2.**—A tangent galvanometer has current constant = 4·1 and is in the primary circuit. The secondary circuit has a resistance of 100 volts and includes a galvanometer whose ballistic constant is $1{\cdot}13 \times 10^{-5}$. The deflection at make in the primary attains the steady value of 40°, and the jump at make or break (opposite ways) is 12 divisions. Find $M$.

Here

$$\delta c = 4{\cdot}1 \times \tan 40^\circ, \quad R = 100, \quad q = 12 \times 1{\cdot}13 \times 10^{-5}.$$

Hence

$$M = 100 \times 12 \times 1{\cdot}13 \times 10^{-5}/4{\cdot}1 \tan 40^\circ = 3{\cdot}94 \times 10^{-3}.$$

**Ex. 3.**—The primary circuit is a solenoid of 1 cm. radius, 12 cm. length, and 200 turns. The secondary is 500 turns wound fairly centrally around the solenoid. Calculate the mutual induction.

Using absolute units, if unit current flows in the solenoid the field inside is nearly uniform and = $4\pi \times 200/12$ (§ 172). This is the number of lines of force per square cm. The area of cross section is $\pi . 1^2 = \pi$, hence the total number of lines penetrating the solenoid is $4\pi^2 \times 200/12$. But each line penetrates the secondary circuit 500 times. Hence the flux in the secondary due to unit current in the primary is $4\pi^2 \times 200 \times 500/12$. By definition, this is the value of $M$ in absolute units. Reducing to Henries by dividing by $10^9$ we get $3{\cdot}29 \times 10^{-4}$.

**Ex. 4.**—Determine a general formula for a standard inductance constructed as above; given the number of turns of primary = $n$, length of solenoid $l$, radius $a$, number of turns of secondary $N$.

Field in solenoid (for unit current) $= 4\pi n/l$, number of lines through it $= \pi a^2 \times 4\pi n/l$, therefore flux through secondary per unit current in primary $= 4\pi^2 a^2 Nn/l$. This is the inductance in absolute units.

**Ex. 5.**—The usual manner of constructing a standard of mutual inductance is this. On a porcelain reel of suitable shape the primary $p$ is wound in two coils of radius $a$, distance apart $a$, as in a Helmholtz galvanometer. The field is then very uniform for a considerable distance around the centre. The secondary $s$ is wound in one coil of radius $b$, considerably less than $a$. If the number of turns of primary and secondary be $n$ and $N$, find $M$.

For a Helmholtz galvanometer the field, or number of lines per unit area, near the centre is (per unit current)

$$2n\pi a^2/\{a^2 + \tfrac{1}{4}a^2\}^{3/2} = 16n\pi/5\sqrt{5}a.$$

The flux per unit current is got by multiplying this by $N\pi b^2$.

Consequently in absolute units

$$M = 16\pi^2 Nnb^2/5\sqrt{5}a.$$

**Ex. 6.**—Calculate in henries the inductance of a standard induction coil constructed as above if there are 900 turns in the primary of radius $a = 3$, and 590 in secondary of radius $b = 2$.

In henries,

$$M = 10^{-9} \times 16\pi^2 \times 900 \times 590 \times 2^2/5\sqrt{5} \times 3 = \cdot 01 \text{ exactly.}$$

*Note.*—In practice, $b$ can be taken as large as $a$. The approximate formula used in Examples 5 and 6 is then replaced by an accurate one calculated by Higher Mathematics.

**Ex. 7.**—Calculate the mutual inductance of the following combination. One of the coils is 225 turns of radius 14 cm., the other is 1000 turns of radius 1 cm. The two coils have the same centre and axis.

**202. The Coefficient of Self-induction** (or the **Inductance**) of a single coil is, in absolute units, the total magnetic flux through that coil, due to unit current in that coil. It is obviously the mutual inductance of that coil with itself. It is measured in henries for practical purposes.

Denote it by $L$. When $c$ is the current, the flux is $F = Lc$. If $c$ should be altering, $\dot{F} = L\dot{c}$. This is numerically

the E.M.F., but the sign should be $-^{ve}$, for induction always opposes the change that produces it (§ 197). Hence $-L\dot{c}$ is the self-induced E.M.F.

**203. Energy of a Current.**—Let the current $c$ increase by the very small quantity $\delta c$ in the short interval $\delta t$; then $\delta c = \dot{c}\delta t$.

The E.M.F. is $L\dot{c}$, hence $L\dot{c}$ is the energy supplied per sec. per unit current. The whole energy supplied per sec. is $Lc\dot{c}$, and $Lc\dot{c}\delta t$ or $Lc\delta c$ is supplied in the interval $\delta t$ considered.

Hence we can write

$$dW = Lcdc,$$

the $d$ denoting (as usual, see § 180) that the changes are indefinitely small. By integration,

$$W = \tfrac{1}{2}Lc^2,$$

no constant being added, for $W = o$ when $c = o$.

This $W$ is the work done in establishing the current, and is not to be confounded with the energy which has to be supplied continually, at the rate $Rc^2$ (§ 107), to maintain the current unaltered.

**204. The Production of Current in a Circuit.**—Let an E.M.F. of amount $E$ be produced in a circuit of resistance and inductance $R$ and $L$. At any time $t$ after make, the self-induced E.M.F. is $-L\dot{c}$ and therefore the resultant E.M.F. is

$$E - L\dot{c} = Rc.$$

Put

$$x = c - E/R,$$

then

$$\dot{x} = \dot{c},$$

$$\therefore \; -L\dot{x} = Rx.$$

This equation is of the type considered in § 182, if $\lambda = R/L$. Its solution is

$$x = x_0 e^{-Rt/L}.$$

But $x_0$ means the initial value of $x$, and $c = o$ initially, therefore $x_0 = -E/R$, and

$$c - \frac{E}{R} = -\frac{E}{R} e^{-Rt/L},$$

$$c = \frac{E}{R}\left\{1 - e^{-Rt/L}\right\}.$$

The same equation results if $R$ and $L$ be the resistance and inductance of a coil whose terminals are suddenly brought to a steady potential difference $E$. At start, $e^{-Rt/L}$ can be put $= 1 - Rt/L$, rejecting squares of small quantities; hence

$$c = Et/L \text{ when } t \text{ is small.}$$

The final state is approached very rapidly if $R/L$ is large, for $Rt/L$ is then considerable even when $t$ is small. The final current ($Rt/L = \infty$) is $E/R$.

If two coils, denoted by suffixes 1 and 2, be in parallel and a potential difference $E$ be applied, then, after a very short time,

$$c_1 = Et/L_1,\ c_2 = Et/L_2,$$

$$\therefore\ c_1/c_2 = L_2/L_1;$$

so that the current divides in inverse ratio to the inductances if it be very transient. Of course, after a considerable time, $c_1/c_2 = R_2/R_1$ as in § 112. This final state is reached very rapidly if $R/L$ be great for both coils. It is obviously sufficient to make $L$ very small; and the coils in a resistance box are always wound so as to diminish $L$ as much as possible.

Since the ordinary law of distribution of currents is not followed, one cannot alter the sensitiveness of a galvanometer by shunts in a known ratio when the galvanometer is used ballistically. Wheatstone Bridge arrangements, and other networks, are frequently used in ballistic work (see §§ 206, 207). But the results are always liable to criticism if due precautions are not taken, such as the use of non-inductive resistances.

If a current $c$ be flowing in a wire whose ends are

suddenly connected so as to reduce the P.D. to zero, the equation is $-L\dot{c} = Rc$; of which the solution is

$$c = c_0 e^{-Rt/L}.$$

**Ex. 1.**—A circuit has resistance $R$, inductance $L$, and contains an E.M.F., $E$ giving a steady current $C = E/R$. Suddenly the resistance is altered to $R'$. Find how the current alters.

Let the current be $c$ at time $t$ after the change. Then

$$E - L\dot{c} = R'c;$$

and putting

$$x = E/R' - c$$

we have, as in above paragraph,

$$x = x_0 e^{-R't/L}.$$

But

$$x_0 = E/R' - C,$$

hence

$$c = \frac{E}{R'}\left\{1 - e^{-R't/L}\right\} + Ce^{-R't/L}.$$

If $t$ is so small that we can neglect $t^2$,

$$c = \frac{Et}{L} + C\left(1 - \frac{R't}{L}\right),$$

$$\therefore\ C - c = Ct(R' - R)/L,$$

so that the current commences to diminish at a rate $L/C(R' - R)$.

**Ex. 2.**—With the notation of last example, find how much extra charge flows round the circuit due to self-induction. $[R' > R.]$

The current ultimately drops from $C$ to $E/R'$ or $CR/R'$. The change of current is therefore

$$C\left(1 - \frac{R}{R'}\right).$$

Hence the self-induced E.M.I. is

$$CL\left(1 - \frac{R}{R'}\right),$$

and the charge = E.M.I./Resistance

$$= \frac{CL}{R'}\left(1 - \frac{R}{R'}\right).$$

**Ex. 3.**—At the centre of a coil of radius 14 and 225 turns is placed a copper ring, radius 1 cm., resistance ·0002 ohm. Calculate

the charge which goes round the ring if a current of 1 amp. is made or broken in the coil.

**Ex. 4.**—The primary of an induction apparatus is a short solenoid, 1000 turns of 1 cm. radius and ·5 ohm resistance. The secondary is 225 turns of 14 cm. radius and 10 ohms resistance. If the terminals of the secondary are joined, determine the charge which flows round if a current of 1 ampere is produced in the primary. Find also the charge which goes round primary for a current of 1 amp. in secondary.

**Ex. 5.**—The arrangement of Ex. 4 is used as a standard of mutual inductance. If one coil can be rotated about a common diameter, show that a rotation $\theta$ multiplies the coefficient of mutual inductance by $\cos\theta$. Find how much the coefficient would be altered by sliding the central primary a distance of 10 cm. perp. to the plane of the secondary.

**205. Comparison of Inductances.**—Let $M_1$ be the mutual inductance of one pair of coils $A$, and $M_2$ of another pair $B$. Arrange the primaries of $A$ and $B$ in series with a battery and a make and break key; then the currents in these primaries are necessarily always the same and the impulsive E.M.F.'s in the secondaries are proportional, for any change of current, to the inductances $M_1$ and $M_2$. These impulsive E.M.F.'s can be compared by Bosscha's or Lumsden's method (§ 131).

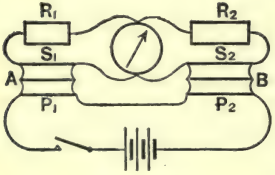

Fig. 107.

The figure may be compared with Fig. 56, and it will be seen that the secondaries $S_1$ and $S_2$ occupy precisely the place of the batteries $B_1$ and $B_2$.

The formula is the same and is proved the same way. If $S_1$ and $S_2$ be the resistance of the secondaries, the impulsive E.M.F.'s (and therefore the inductances $M_1$ and $M_2$) are in ratio $R_1 + S_1 : R_2 + S_2$. If other resistances $R_1'$, $R_2'$, also give a balance,

$$M_1 : M_2 = R_1 + S_1 : R_2 + S_2 = R_1' + S_1 : R_2' + S_2 ;$$

hence

$$M_1 : M_2 = R_1' - R_1 : R_2' - R_2.$$

Another method is to put resistances $r_1$ and $r_2$ in series with the primaries $P_1$ and $P_2$ and arrange these in parallel with a battery and key in the circuit (Fig. 108). The secondaries are arranged in series with the ballistic galvanometer. Adjust $r_1$ and $r_2$ for a balance. When this is attained, the currents in $P_1$ and $P_2$ are inversely as $P_1 + r_1$ and $P_2 + r_2$; and the impulsive E.M.F.'s in $S_1$ and $S_2$ are therefore in the ratio $M_1/(P_1 + r_1) : M_2/(P_2 + r_2)$. The ballistic galvanometer is unaffected if

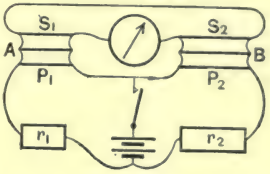

Fig. 108.

$$M_1 : M_2 = P_1 + r_1 : P_2 + r_2.$$

A second balance is obtained with other resistances $r_1'$ and $r_2'$, then

$$M_1 : M_2 = P_1 + r_1' : P_2 + r_2',$$
$$\therefore\ M_1 : M_2 = r_1' - r_1 : r_2' - r_2.$$

A comparison of the figures will show that the two methods differ only in having the battery and galvanometer interchanged.

In both methods alike, it is quite possible to arrange the coils in such a way that no balance can be obtained because the two inductances assist one another. If this fault is suspected, reverse the connections either to one primary or one secondary. Precautions ought to be taken that the magnetic field of neither primary affects the other secondary; and that these magnetic fields do not affect the galvanometer.

**206. Comparison of Mutual with Self Inductance.** —To compare $M$, the mutual inductance of two coils $S$ and its primary, with $L$, the self-inductance of $S$. Connect as in figure. In the first place the three resistance boxes $P$, $Q$, $R$ may be arranged with $S$ in an ordinary Wheatstone Net, and $P$ and $T$ may be omitted. There must be two keys, the battery key and galvanometer key, B.K. and G.K.

in figure. Alter $P$ and $Q$ (and $R$ if desired) to get an exact balance; testing always by first pressing down B.K., and not pressing G.K. until all transient induced currents are exhausted. The necessary time is generally a minute fraction of a second; so that the keys are pressed almost simultaneously, but B.K. must have priority. The balance must be exact. As $P$, $Q$, $R$ are generally boxes in integral ohms, the nearest approximate balance must be corrected by adding a piece of fine wire to any convenient resistance, preferably to the largest one.

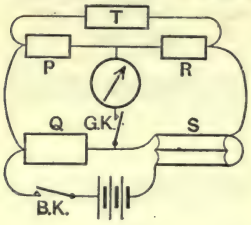

Fig. 109.

When the balance is perfect, it will not be affected in any way by joining on the resistance box $T$.

$T$ should now be joined in (unless this was done at start), and the balance should be tested by giving the priority to G.K. instead of B.K. as before. The galvanometer needle cannot be steadily deflected, for the balance is bound to be perfect when the transient currents are exhausted. But it may be jerked by the transient currents. If all is well, the jump is one way when $T = \infty$ [*i.e.* $T$ disconnected] and the other way when $T = o$. If this is not the case, reverse the connections of $P$ so that the primary current goes through it the opposite way. When this is the case, a balance is only reached when the impulsive E.M.F. in $S$ is zero.

Let $C$ be the final current through $P$. The E.M.I. in $S$ due to mutual inductance is $\pm$ M.C. The current $C$ in $P$ is finally divided between the three resistances $Q + S$, $P + R$, and $T$; so that the final current in $S$ is

$$\frac{C}{Q+S} \Big/ \left\{ \frac{1}{Q+S} + \frac{1}{P+R} + \frac{1}{T} \right\} = C' \text{ say.}$$

Consequently the E.M.I. in $S$ due to self-inductance is $\pm LC'$. For a balance, they are contrary ways, and

$$LC' = MC,$$

$$\therefore \frac{L}{M} = \frac{C}{C'} = (Q+S)\left\{\frac{1}{Q+S} + \frac{1}{P+R} + \frac{1}{T}\right\},$$

$$\therefore \frac{L}{M} = 1 + \frac{Q+S}{P+R} + \frac{Q+S}{T}.$$

But $PS = QR$ for the preliminary balance; therefore eliminating $S$,

$$\frac{L}{M} = 1 + \frac{Q}{P} + \frac{Q(P+R)}{PT}.$$

The method only applies, therefore, if $L > M$; and it will be impossible to get the second balance by merely altering $T$ unless $L/M > 1 + Q/P$. If no second balance can be got, even by reversing the connections of the primary, a new first balance must be obtained with a large $P$ and small $Q$. The difficulty seldom occurs, for generally $L$ is much larger than $M$.

**207. Comparison of Self-inductions.**—To compare the self-inductions $L_1$ and $L_2$ of two coils $U$ and $V$.

The connections (Fig. 110) look complicated, but are based on the ordinary Wheatstone net. If the resistance box $T$ has all its plugs in, $T$ can be regarded as a mere point, and $R$, $V$, $U$, $S$ form the four arms of a bridge. A perfect balance is obtained, the battery keys being always put down first. We then have

$$R/U = V/S.$$

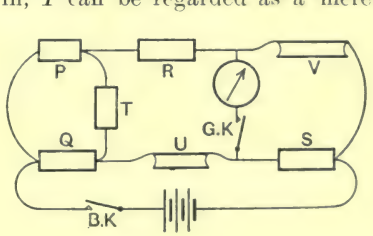


Fig. 110.

Next make $T$ infinite. Alter $P$ and $Q$ to obtain a perfect balance (with the battery key down first). Since $P + R$, $V$, $Q + U$, $S$ are four arms of a bridge,

$$(P+R)/(Q+U) = V/S.$$

Therefore $P/Q = R/U = V/S$;

so that $U = QR/P$, $V = PS/Q$.

The balance (for steady currents) will now be perfect whatever $T$ may be; since no current goes through $T$.

Finally, get a balance for transient currents, the galvanometer key being put down first in every test and $T$ being altered until there is no jerk.

It is evident that the final steady currents through $PRV$ and $QUS$ are in ratio

$$Q + U + S : P + R + V = Q : P.$$

Hence the impulsive E.M.F.'s in $V$ and $U$ are in ratio $L_2Q : L_1P$.

But, taking the case of currents at break for simplicity, the theory of Bosscha's or Lumsden's method shows that these impulsive E.M.F.'s are in the ratio of $V + S$ to $R + X + U$, where $X$ is the resistance of $T$ and $P + Q$ in parallel.

Hence

$$\frac{L_2Q}{L_1P} = \frac{V+S}{R+U+X} = \frac{V+S}{R+U+T(P+Q)/(T+P+Q)},$$

$$= \frac{PS/Q+S}{R+QR/P+T(P+Q)/(T+P+Q)},$$

$$= \frac{(P+Q)S/Q}{(P+Q)R/P+T(P+Q)/(T+P+Q)},$$

$$= \frac{PS(T+P+Q)}{Q\{RT+RP+RQ+TP\}},$$

$$L_1/L_2 = Q^2\{PR+QR+PT+RT\}/P^2S(T+P+Q).$$

Obviously this formula enables one to find the self-inductance of any coil if one has a standard of self-inductance. The precautions needed in carrying out this experiment are that the current balance must be exact, that the resistances $P$, $Q$, etc., are non-inductive; and that $U$ and $V$ be so placed that they neither influence the galvanometer at make nor have a mutual inductance.

**208. Direct Measurement of a Self-induction.**—Let $R$ be the resistance, and $L$ the inductance, of a coil. It is arranged with adjustable resistances $P$, $Q$, $S$, and a battery and galvanometer, so as to form a Wheatstone

Bridge as in Fig. 111. Another box $r$ is inserted with $R$; it has all its plugs in at first, and therefore adds no resistance.

Get a balance (exact) when the battery key is put down before the galvanometer key; then $P/Q = R/S$. Put down the galvanometer key first, and then the battery key, and notice the jerk $x$. If $q$ be the ballistic constant of the galvanometer, $qx$ is the charge which has been sent through it. If $C$ be the final current in the coil $R$, $CL$ is the E.M.I. which causes the charge $qx$ to flow. The ratio $qx/CL$ can be calculated according to the theory explained for steady currents in § 53, III. In fact, replacing $F, G, H, L, M, N$ by $Q, P, G, S, B, R$ ($B$ and $G$ being resistances of battery and galvanometer) and $E, x - y$ by $CL, qx$, we get

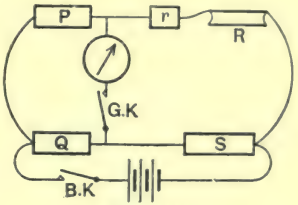

Fig. 111.

$$qx = CLP/(QR + PG + QG + PQ),$$

whence $P$ is found. But as this necessitates using a formula which is neither easy to remember nor easy to find, one generally proceeds as follows. The small resistance $r$ is introduced, and the steady deflection $y$ due to the disturbance of the balance is observed. If $c$ be the current constant of the galvanometer, $cy$ is the current through it. The resistance $R$ carries a current $C$ which will hardly alter for the small change of $R$ into $R + r$. This alteration is equivalent to inserting an E.M.F. (backward) of amount $rC$; so that the current $cy$ is due to the E.M.F. $rC$ in the same network in which a charge $qx$ is due to the E.M.I. $LC$. Hence

$$qx/cy = LC/rC = L/r.$$

But if $T$ be the period of the needle, $q/c = T/2\pi$ (§ 185). Hence

$$L = Trx/2\pi y.$$

Instead of increasing $R$ by $r$, one may diminish $S$ by $s$

This has ultimately the same effect if $r/R = s/S$, because proportional charges of like sign would leave the balance unaltered and not affect the galvanometer. Or one can diminish $P$, or increase $Q$, by $p$, $q$. Then calculate $r$ from

$$r/R = \text{either } s/S \text{ or } p/P \text{ or } q/Q$$

and use the above formula.

**209. Comparison of Inductance and Capacity.**—Let $L$ be the self-inductance of a part of a circuit, the rest of which is non-inductive. Let $r$ be the resistance of any portion of the circuit, and let it be put in parallel with a condenser of capacity $K$ and with a key. Let $R$ be the whole resistance of the circuit, and $C$ the final current. The E.M.I. of the coil is $-LC$.

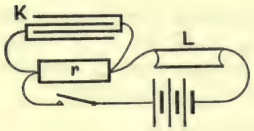

Fig. 112.

Evidently the condenser is finally charged to potential difference $rC$, therefore the charge which enters it is $KrC$. If $q$ be the extra charge which traverses $r$ at make or break, $q + KrC$ must traverse the resistance $R - r$. Put

$$q + KrC = Q.$$

Now Kirchoff's law for any mesh, that the sum of products of current and resistance all round the mesh is equal to the sum of E.M.F.'s round the mesh, is, by integration, also true for charges and E.M.I.'s.

Therefore

$$qr + Q(R - r) = -LC.$$

But

$$q = Q - KrC,$$

$$\therefore QR - Kr^2C = -LC,$$

so that

$$Q = o \text{ if } L = Kr^2.$$

Hence if this condition is fulfilled there ceases to be any extra current except in the section $r$ and the leads of the

condenser. A ballistic galvanometer anywhere else in the circuit would be unaffected.

**Ex. 1.**—The self-induction of a coil is ·012 henry. Find what resistance, in parallel with a one-third microfarad, will apparently annihilate this inductance.

We have

$$L = Kr^2\,;\ L = 1{\cdot}2 \times 10^{-2},\ K = 10^{-6}/3\,;$$

hence

$$r^2 = 3{\cdot}6 \times 10^4\,;\ r = 190.$$

**Ex. 2.**—Estimate the self-induction of a solenoid of 1 cm. radius, 12 cm. length, and 200 turns. What difference would it make if there were 500 turns?

As in Ex. 3 of § 201, the total number of lines through the first coil is $4\pi^2 \times 200/12$. Each line penetrates the first coil 200 times, therefore its self-induction

$$L = 200 \times 4\pi^2 \times 200/12 \times 10^9 = 1{\cdot}32 \times 10^{-4}.$$

With 500 turns,

$$L' = 500 \times 4\pi^2 \times 500/12 \times 10^9 = 8{\cdot}22 \times 10^{-4}.$$

It is easily seen that coils of the same dimensions but different numbers of turns have inductances proportional to the square of the number of turns.

**Ex. 3.**—If it were possible to wind two coils of $n$ and $N$ turns so close to one another that any magnetic field enclosed by either was necessarily enclosed by the other also, show that their mutual inductances $l$ and $L$, and self-inductance $M$, are in the ratios $n^2 : N^2 : nN$, and that $Ll = M^2$.

If unit current be produced in the first coil, let $f$ be the number of lines enclosed by the first coil and therefore by the second. In estimating the magnetic flux, each line must be counted as penetrating the first coil $n$ times and the second $N$ times. But

$l$ = flux through first coil due to unit current in that coil (Def. § 202)

$M$ = flux through second due to unit current in first (Def. § 200)

therefore

$$l = nf,\ M = Nf\,;\ \text{and}\ l : M = n : N.$$

Interchanging the coils (which does not alter $M$)

$$L : M = N : n.$$

Hence

$$L : M : l = N^2 : Nn : n^2,$$

and

$$Ll = M^2.$$

**210. Energy of Two Circuits.**—Let the circuits have mutual inductance $M$ and self-inductances $L$ and $l$. Let them carry currents $C$ and $c$.

The energy of the first coil regarded as a system in itself is $\frac{1}{2}LC^2$. The energy of the second is $\frac{1}{2}lc^2$. The mutual energy is $MCc$. Hence the total

$$= W = \tfrac{1}{2}LC^2 + MCc + \tfrac{1}{2}lc^2$$
$$= \tfrac{1}{2}\{LC^2 + 2MCc + lc^2\}.$$

Here $L$ and $l$ are essentially positive; but by reversing the connections of either coil we can alter the sign of $M$, so $M$ is doubtful. But it is physically impossible that $W$ should be negative, therefore the quantity $LC^2 + 2MCc + lc^2$ is positive whatever $C$ and $c$ may be. Hence, for every arrangement of two coils,

$$Ll > M^2.$$

In Ex. 3 of § 208 we have met with a hypothetical case in which $Ll = M^2$. Though this is mathematically impossible, it can be approached so closely that we can treat it as an ordinary case in electrical engineering, where of course high approximations are treated as absolute results.

The expression for the energy leads to various consequences given below as examples.

**Ex. 1.**—What is the self-induction of two coils connected (either way) in series? $L$, $l$ and $M$ are supposed given.

Here $C = \pm c$. With upper sign, $W = \frac{1}{2}(L + 2M + l)\,c^2$, hence the self-induction is $L + l + 2M$. With lower sign, the self-induction is $L + l - 2M$.

Therefore we can find Mutual Induction by two measurements of Self-induction. This is not, perhaps, particularly useful, because it is easier to find $M$ direct.

**Ex. 2.**—What is the self-induction of two coils (of resistances $R$ and $r$) connected in parallel?

If $C$ be the total current, the currents in the coils are $Cr/(R + r)$, $CR/(R + r)$. Hence

$$W = \tfrac{1}{2}\{Lr^2 \pm 2MrR + lR^2\}C^2/(R + r)^2,$$

and the self-induction is

$$\frac{Lr^2 \pm 2\,MrR + lR^2}{(R + r)^2}.$$

**Ex. 3.**—If two coils be in parallel, find the ratio of resistances so that the self-induction be a minimum.

Let

$$x = r/(R + r), \text{ then } 1 - x = R/(R + r)$$

and the induction (taking $M$ negative)

$$= Lx^2 - 2Mx(1 - x) + l(1 - x)^2,$$
$$= l - 2(M + l)x + (l + 2M + L)x^2,$$
$$= \frac{\{(l + 2M + L)x - (M + l)\}^2 + Ll - M^2}{l + 2M + L}.$$

This is least when the squared term is zero,

$$\therefore \quad x = \frac{r}{R + r} = \frac{L + M}{l + 2M + L}$$

and

$$\frac{R}{R + r} = \frac{L + M}{l + 2M + L},$$

so that

$$R : r = L + M : M + l.$$

**Ex. 4.**—If two coils be separate, it is always possible to arrange them so that $M = o$. Show that their self-inductances can then be compared by arranging them in parallel, each in series with a resistance box, and adjusting until their combined self-induction is a minimum.

Obviously putting $M = o$ in the above, $R : r = L : l$; hence $L : l$ is found.

**211. Source of the Field Energy.**—Let $E$ be E.M.F. of the battery, and $R$ be resistance. At make, the current alters from $o$ to $C$; therefore the E.M.I. is $LC$ and the "extra charge" is $LC/R$. As this charge goes through the battery it produces electrolysis, and the chemical energy supplied $= ELC/R = LC^2$, since $C = E/R$. Consequently the extra energy supplied by the battery is $LC^2$; half of this establishes the energy of magnetic field and the other half becomes heat.

**Ex. 1.**—Two identical flat coils in series with a battery are gradually moved, from a distance, to the nearest possible coincidence (with the currents going opposite ways). The energy of the field was $LC^2$ originally; and finally is practically zero, since the fields of the coils annul one another at all points moderately distant. What becomes of the lost energy?

In the position of coincidence, $M = -L$ since the flux due to either coil is the same in the other coil as in itself. Hence $LC$ is the E.M.I. due to mutual inductance in each coil; and if $R$ be the total resistance of circuit, the extra current carries a charge $2LC/R$. The induced current is the same way as the original one, therefore this charge takes energy $2ELC/R$ from the battery; this $= 2LC^2$. The currents repel one another, so mechanical work $MC^2 = LC^2$ has to be done to put them into the position required. The lost energy of field is $LC^2$. So there is a total energy $4LC^2$ apparently wasted, but actually spent in heating the wires.

**Ex. 2.**—What difference would be made if the coils had been moved to the nearest coincidence with the coils the same way round?

The energy of the field changes from $LC^2$ to $2LC^2$; the battery supplies less energy to the extent $2LC^2$; and the mechanical attraction provided $LC^2$ to the motive apparatus, therefore $LC^2$ of potential energy is lost. The total loss is $4LC^2$; and the wires are less heated (to this extent) than they would have been had no motion taken place.

**Ex. 3.**—A primary current is arranged to carry a current $C$, as in Fig. 106. The resistance of the secondary is $R$ and the mutual induction is $M$. How much heat is developed in the secondary at make or break of the primary? What is the source (in each case) of the energy that appears as heat?

**Ex. 4.**—With the arrangement of Fig. 112, if $L = Kr^2$, show that the work done by the battery during make is twice as great as if the condenser were absent; and that the energy of the electromagnetic field is equal to that of the charged condenser.

**Ex. 5.**—Find in Joules and in calories the energy needed to establish a current of ·1 ampere in a coil of self-inductance ·05 henry.

**212. Alternate E.M.F.**—The general equation connecting E.M.F. and current in a circuit is

$$E = L\dot{C} + RC \qquad (\S 99)$$

Cases are often met with (see Ex. 1 below and § 236) where the E.M.F. alters harmonically with the time, and we can therefore write

$$E = E_0 \cos nt,$$

the period being

$$T = 2\pi/n.$$

Assume as a trial solution $C = C_0 \cos(nt - \alpha)$, then $\dot{C}$ or $dC/dt = -nC_0 \sin(nt - \alpha)$, hence

$$E_0 \cos nt = C_0 \{R \cos(nt - \alpha) - nL \sin(nt - \alpha)\}.$$

But by trigonometry

$$\begin{aligned} E_0 \cos nt &= E_0 \cos\{(nt - \alpha) + \alpha\} \\ &= E_0 \cos(nt - \alpha) \cos\alpha - E_0 \sin(nt - \alpha) \sin\alpha, \end{aligned}$$

hence, equating coefficients,

$$E_0 \cos\alpha = C_0 R, \quad E_0 \sin\alpha = C_0 nL,$$

therefore

$$\tan\alpha = nL/R$$

and

$$E_0^2 = C_0^2(R^2 + n^2L^2).$$

The maximum current $C_0$ is connected with the maximum E.M.F. $E_0$ by the equation

$$C_0 = E_0/\sqrt{R^2 + n^2L^2}.$$

**213. Lag and Impedance. Choking Coil.**—The above equation resembles Ohm's law, $C = E/R$; therefore $\sqrt{R^2 + n^2L^2}$ is the "apparent resistance." It is called the Impedance.

The maximum current $C_0$ occurs when $nt = \alpha, \alpha + 2\pi, \alpha + 4\pi$, etc.; and the maximum E.M.F. when $nt = o, 2\pi, 4\pi$, etc. Consequently the maximum current is later than the maximum E.M.F. by a time $\alpha/n$, where $\alpha$ is an acute angle. This angle is given by $\tan\alpha = nL/R$. This acute angle is called the Lag. The frequency of the current is, of course, $= 1/T = n/2\pi$.

The rate at which energy is being supplied to the system at any instant is $EC$, which

$$\begin{aligned} &= E_0 \cos nt \,.\, C_0 \cos(nt - \alpha), \\ &= E_0 C_0 \{\cos^2 nt \cos\alpha + \sin nt \cos nt \sin\alpha\}. \end{aligned}$$

Now the average value of $\cos^2 nt$, for a very long period, is $1/2$; and the average of $\sin nt \cos nt$ or $\frac{1}{2}\sin 2nt$ is zero, for $\sin 2nt$ is a periodic function as often $+$ as $-$. There-

fore the energy supplied is at the average rate (see § 181)

$$= \tfrac{1}{2} E_0 C_0 \cos \alpha = \tfrac{1}{2} R C^2{}_0,$$

$$= R \times \text{average value of } C^2$$

or $$= \tfrac{1}{2} R E^2{}_0/(R^2 + n^2 L^2).$$

If $R$ is very small but $nL$ large, the current will be small and the waste of current energy will be very small also. A coil of this kind is called a Choking Coil.

**Ex. 1.**—A flat coil of $N$ turns and area $A$ (as in Ex. 2 of § 197) rotates about a vertical axis with angular velocity $\omega$. Its ends are connected. Its resistance is $R$ and inductance $L$. Find how the current alters with the time.

When $\theta$ is the angle rotated through, the component of $H$ perpendicular to the coil $= H \sin \theta$, and the total flux $F = NAH \sin \theta$.

The E.M.F. is $E = \dot{F} = NAH \cos \theta \,.\, \dot{\theta}$.

But we can take $\theta = \omega t$ and $\dot{\theta} = \omega$; hence

$$E = NAH\omega \cos \omega t = E_0 \cos \omega t,$$

where $$E_0 = NAH\omega.$$

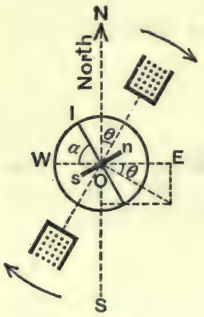

Fig. 113.

Hence the current

$$= C_0 \cos (\omega t - \alpha)$$

where

$$C_0 = E_0/\sqrt{R^2 + \omega^2 L^2}$$

and

$$\tan \alpha = L\omega/R.$$

We may notice two cases. At low speeds, or if $L/R$ is small, $\tan \alpha$ is small, and approximately $\alpha = 0$,

$$C_0 = E_0 \cos \omega t/\mathrm{R} = NAH\omega \cos \omega t/R.$$

The current is therefore a maximum when $\omega t = 0$ or when the coil is in the plane of the magnetic meridian. At high speeds, or if $L/R$ is large, $\tan \alpha$ is large, $\alpha$ approaches the value 90°, and the current is a maximum when $\omega t =$ nearly 90°, or the coil is in the east and west position.

**Ex. 2.**—Find the effect of a circular coil of radius $r$, rotating as above, on a compass needle supported at rest at the centre of the coil.

When $\theta = \omega t$ and the current is $C = C_0 \cos (\omega t - \alpha)$, the field $X$ due to $C$ is

$$X = 2\pi CN/r \qquad [\S\ 161, \text{Ex. } 1.]$$

$$= 2\pi NC_0 \cos (\omega t - \alpha)/r,$$

and it makes an angle $\theta = \omega t$ with the magnetic east and west line. If the rotation be in the direction indicated in Fig. 113, the eastward component is

$$X \cos \omega t = 2\pi N C_0 \cos \omega t \cos (\omega t - \alpha)/r$$
$$= 2\pi N C_0 \{\cos^2 \omega t \cos \alpha + \sin \omega t \cos \omega t \sin \alpha\}/r ;$$

and the southward component is

$$X \sin \omega t = 2\pi N C_0 \sin \omega t \cos (\omega t - \alpha)/r$$
$$= 2\pi N C_0 \{\sin \omega t \cos \omega t \cos \alpha + \sin^2 \omega t \sin \alpha\}/r.$$

The average values are

$$\text{Eastward, } X_1 = \pi N C_0 \cos \alpha/r = \pi N V_0 R/(R^2 + L^2\omega^2)r,$$
$$\text{Southward, } X_2 = \pi N C_0 \sin \alpha/r = \pi N V_0 \omega L/(R^2 + L^2\omega^2)r.$$

The compass needle will behave as if affected by the earth's field $H$ northward, and by the above average values. It will deflect an angle $NOn = \phi$ from the north, given by

$$\tan \phi = X_1/(H - X_2).$$

Replacing $V_0$ by its value $NAHw$ we get, cancelling $H$ and simplifying,

$$\text{Tan } \phi = \frac{\pi N^2 A \omega R}{(R^2 + L^2\omega^2)r - \pi N^2 A L \omega^2}.$$

It will be sufficient to suppose $L$ small. Then approximately

$$\tan \phi = \frac{\pi N^2 A \omega}{Rr}.$$

But $A = \pi r^2$, hence $\tan \phi = \pi^2 N^2 r \omega / R$. An observation of $\phi$ enables one therefore to calculate $R$ in absolute units. A correction for the value of $L$ ought, of course, to be applied. $L$ is calculable mathematically from the number of turns, or can be eliminated by a series of observations at different speeds ; and the effect of $L$ is very small if the speed is small. Since $H$ does not appear in the final result, the accurate determination of $H$ is not important.

This experiment is the British Association method for standardising the ohm. The ohm based on this method was called the B.A. unit. The actual B.A. experiment was considerably more complicated ; for instance the rotating coil was not a flat circular one, but a pair of circles arranged as in a Helmholtz galvanometer. A number of subsidiary experiments were performed at the same time, in which the Clark Cell was standardised and the silver deposited electrochemically per coulomb was determined.

# CHAPTER XII.

## PERMEABILITY.

**214. Field and Induction in Magnetic Shell.**—Consider a uniform plane magnetic shell (§ 166) of strength $S$, whose thickness $x$, though finite, is very small compared to its area. Since $S$ is the magnetic moment per unit area, the magnetic moment per unit volume will be $I$ where

$$I = S/x.$$

The surface density $\pm\sigma$ of $N$ magnetism on the faces = pole strength per unit area $= S/x = I$.

Let $X$ be the normal component of that magnetic field which is due to external magnets other than the shell.

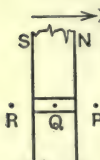

Fig. 114.

The field due to a flat plate of density $\sigma = \pm I$, close to the plate, is $\pm 2\pi I$ (§ 34). The face $N$ which has north magnetism exerts force $2\pi I$ to right on a unit $N$ pole at $P$, and to left at $Q$ and $R$. Similarly $S$ exerts $2\pi I$ to left at $P$ and $Q$ and to right at $R$. Hence the total influence of the shell is zero at $P$ and $R$, and $4\pi I$ to left at $Q$. To avoid the complication of having $Q$ within the actual material of the shell, it may be considered to be in a tunnel so narrow that the influence of the ends of the tunnel is negligible.

The normal fields at $P,R,Q$ are $X,X$ and $X - 4\pi I$ respectively. Although $Q$ is not literally within the shell, we shall define the field $Z$ within the shell (neglecting the tangential component) as the field at $Q$, and we therefore have

$$Z = X - 4\pi I.$$

The fields at $P$ and $R$ are the same, $= X$. There are $X$ lines of force per unit area reaching the face $R$, and

leaving the face $N$. It is reasonable to imagine the lines continuous, and then there would be $X$ lines per unit area within the shell. They are here not called lines of force, but **lines of induction.** The induction $X$ is the number of lines per unit area taken perpendicular to the direction of the lines. Within the shell the field or force is $Z$, and we say there are $Z$ **lines of force** per unit area.

If there were a tangential component $Y$ of the external field, we should simply add a tangential component $Y$ both to the force and induction as above described.

**215. Field and Induction within any Magnet.**—At any point $Q$ within a magnet, let $I$ be the intensity of magnetisation. That is, if any small volume $V$ around $Q$ be considered, its magnetic moment is proportional to $V$ and $= VI$, and points in a certain direction called the direction of $I$. We can draw two parallel planes, perpendicular to $I$, and one each side of $Q$. The distance between can be as small as we like; so that we have traced in the magnet a magnetic shell.

The field at $Q$ is defined as the field in a small tunnel constructed, as in § 214, normal to this shell. Hence the tunnel has its axis parallel to the direction of $I$. This field $H$ may have a tangential and normal component $Z$ and $Y$ as above.

The induction at $Q$ would be, for a shell, the field just outside the shell in air (§ 214). Its components are $X$ and $Y$. If we consider the magnet as constructed of shells whose faces are perpendicular to $SN$, the induction in each is the field which we should get if we left an air space between it and the next. This is the field which would exist in a fissure or crevasse of small but finite width. It is called $B$. Now the tangential components of $B$ and $H$ are the same, and the normal components satisfy

$$X = Z + 4\pi I.$$

Hence we can write

$$B = H + 4\pi I$$

if we remember that the $+$ only denotes addition when $H$ and $4\pi I$ are in the same line, but denotes composition or vector addition in other cases.

**Example.**—A long bar magnetised longitudinally with uniform intensity $I$ has cross section $A$. If it were sawn through transversely, find the force which one portion of it would exert on the other portion.

The surface $Q$ is uniformly spread with magnetic matter of surface density $I$. Consequently the attraction it would exert on unit $S$ pole close to it is

$$2\pi I. \qquad (\S\ 34)$$

But the surface $R$ carries $AI$ units of south polar magnetism. Hence the resultant attraction

$$= 2\pi I^2 A.$$

There is thus an extra magnetic tension of value $2\pi I^2$ per square centimetre.

**216. Definition of Field and Induction.**—The field $H$, at any point $O$ within a magnet, is the field which would be observed in air* in a small tunnel constructed around that point; provided that the axis of the tunnel is in the direction of the magnetisation, and the cross section of the tunnel is indefinitely small compared to its length.

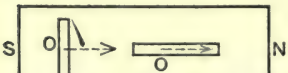

Fig. 115.

The induction $B$, at any point within a magnet, is the field which would be observed in air in a small fissure or crevasse whose plane faces are perpendicular to the direction of magnetisation and have diameters and areas infinitely large compared to the distance between them. In Fig. 115 the tunnel is shown to right, and the crevasse to left.

**217. Induced Magnetisation.**—For a large number of substances the following law holds accurately. The exceptions are the most strongly magnetic substances, including iron, nickel and cobalt. For these, the law is obeyed in very weak fields with a certain degree of accuracy.

The total induction ($B$) is in the same direction as the total field ($H$) and proportional to it.

* When great accuracy is sought, vacuum is substituted for air in these definitions.

The constant ratio of induction to field ($B/H$) is called the Permeability ($\mu$), and is positive. Hence

$$B = \mu H.$$

Combine this formula with $B = H + 4\pi I$, and we have

$$4\pi I = (\mu - 1)H.$$
$$\therefore\ I = kH,$$

where $k$ is a constant called the Susceptibility, and connected with $\mu$ by the equation

$$4\pi k = \mu - 1, \quad \mu = 1 + 4\pi k.$$

**218. Paramagnetism and Diamagnetism.**—If $\mu > 1$, $k$ is $+^{ve}$; and the substance is said to be paramagnetic. Examples are iron, nickel, cobalt, ferric chloride, liquid oxygen, etc. If $\mu = 1$, $k = o$; and the substance is indifferent. Gases should be compared with a vacuum; not with air (see footnote, § 212). Any rare gas, as normal air, is almost indifferent.

If $\mu < 1$, $k$ is negative. The substance is called Diamagnetic. Examples are bismuth, and liquid nitrogen.

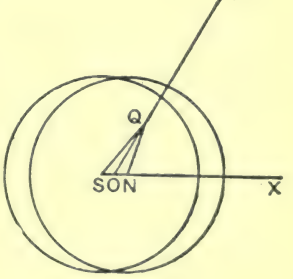

Fig. 116.

**219. Uniformly Magnetised Sphere.**—The magnetisation of the sphere may be considered due to a large number of doublets of common pole-strength $\pm m$ and length $l$, all in the same direction $OX$, and uniformly distributed. If $n$ be the number per unit volume, the magnetic moment per unit volume is $nml$, hence

$$nml = I.$$

If $a$ be the radius of the sphere, its whole magnetic moment is

$$M = \tfrac{4}{3}\pi a^3 I.$$

The $N$ poles are individually at a constant distance $l$, parallel to $OX$, from their corresponding $S$ poles, hence they can be regarded as uniformly distributed through a sphere of centre $N$, while the $S$ poles are uniformly distributed through one of centre $S$, where $SN = l$. There are $n$ $N$ poles (or $S$ ones) per unit area, hence the density of each magnetisation is $nm$. The sphere of $N$ magnetisation acts at all external points as if its magnetic matter were concentrated at $N$. It therefore is equivalent to a single pole at $N$ of strength $K = \frac{4}{3}\pi a^3 nm$. Similarly the sphere of $S$ magnetisation acts like $-K$ at $S$. Hence the whole magnetised sphere acts like a doublet situated at $C$, of strength $Kl = \frac{4}{3}\pi a^3 nml = \frac{4}{3}\pi a^3 I$. If $P$ be an external point, the field at $P$ is the resultant of $K/NP^2$ along $NP$, $K/SP^2$ along $PS$; and can be found as in § 150.

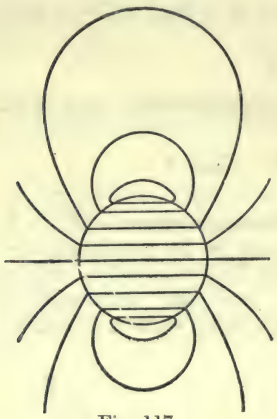

Fig. 117.

If $Q$ be an internal point, the sphere of centre $N$ composed of north magnetisation can be regarded as a sphere of radius $NQ$, mass $\frac{4}{3}\pi NQ^3.nm$; and a shell whose external and internal radii are $NQ$ and $a$. The latter exerts no force; the former exerts $\frac{4}{3}\pi NQ^3.nm/NQ^2 = \frac{4}{3}\pi nm.NQ$ along $NQ$. The sphere of south magnetisation has a field $\frac{4}{3}\pi nm.QS$ along $QS$. The resultant of these is parallel to $NS$ and $= \frac{4\pi}{3}.nm.NS = \frac{4}{3}\pi I$. Strictly it is negative, being in the opposite direction to $I$. The induction

$$B = 4\pi I - \frac{4}{3}\pi I = \frac{8}{3}\pi I.$$

The lines of induction (external and internal) are shown in Fig. 117.

**220. Magnetisation Induced on a Sphere in a Uniform Field.**—Let $X$ be the strength of the inducing field. Assume that the induced intensity of magnetisation is equal to $I$ and in the direction of $X$.

The field within the sphere due to the sphere itself is $-\frac{4}{3}\pi I$, hence the total field is

$$H = X - \tfrac{4}{3}\pi I.$$

But

$$B = H + 4\pi I \qquad (\S\ 215),$$
$$\therefore\ B = X + \tfrac{8}{3}\pi I.$$

And

$$B = \mu H.$$

Hence

$$X + \tfrac{8}{3}\pi I = \mu(X - \tfrac{4}{3}\pi I),$$
$$\therefore\ I = \frac{3}{4\pi}\frac{\mu - 1}{\mu + 2}X,$$
$$B = \frac{3\mu X}{\mu + 2},\quad H = \frac{3X}{\mu + 2}.$$

[*Note.*—In the limit when $\mu$ is $\infty$, $I = 3X/4\pi$, and $B = 3X$. These are independent of $\mu$ and very small compared to the corresponding quantities for a bar; see Ex. 1 below.]

**Ex. 1.**—A very long unmagnetised bar of permeability $\mu$ is placed longitudinally in a weak field of strength $X$. Assuming that it is uniformly magnetised, by induction, to intensity $I$, find the ratio of $X$ to $I$.

In a long bar, uniformly and weakly magnetised, the field due to the bar itself can be disregarded for nearly the whole volume of the bar. Hence the actual field within most of the bar is the field due to causes other than the bar's magnetisation, and therefore is $H = X$.

But

$$B = \mu H,$$
$$\therefore\ B = \mu X,$$

and

$$I = \frac{B - H}{4\pi} = \frac{\mu - 1}{4\pi}X.$$

**Ex. 2.**—If the bar be long and narrow, and be placed obliquely at slope $\theta$ to the direction of the field, find the couple that acts on it if $\mu - 1$ be not small.

The longitudinal field is $H\cos\theta$. The transverse field is $H\sin\theta$. The former produces longitudinal magnetisation of intensity $I$

$(\mu - 1)H \cos \theta/4\pi$ as above; the latter we shall disregard. If $V$ be the volume, then the magnetic moment is

$$M = VI = V(\mu - 1)H \cos \theta/4\pi,$$

and the couple is (§ 143)

$$\begin{aligned} MH \sin \theta &= V(\mu - 1)H^2 \sin \theta \cos \theta/4\pi, \\ &= VkH^2 \sin \theta \cos \theta. \end{aligned}$$

For paramagnetic bodies $k$ is positive, and this couple tends to diminish $\theta$ and draw the bar into a longitudinal direction.

**Ex. 3.**—To find the period of small oscillation of a long paramagnetic bar in a magnetic field.

Let $K$ be its moment of inertia. The couple, if $\theta$ be small, $= L\theta$ where $L = V(\mu - 1)H^2/4\pi$. The period (§ 154) is $T = 2\pi\sqrt{K/L}$.

If $d$ be the density, the mass is $Vd$, and the moment of inertia is (nearly)

$$K = Vdl^2/12,$$

$$\therefore \quad K/L = \pi l^2 d/3(\mu - 1)H^2,$$

and the period

$$\begin{aligned} &= 2\pi l \sqrt{\pi d/3(\mu - 1)} \div H \\ &= \pi l \sqrt{d}/H \sqrt{3k}, \text{ since } \mu - 1 = 4\pi k. \end{aligned}$$

**Ex. 4.**—A large number of small spheres, each of great permeability ($\mu = \infty$), are distributed evenly through a nonmagnetic matrix. To find the apparent permeability of the mixture.

Let $X$ be the uniform field which induces magnetisation. If we can neglect the effect of the spheres on one another, the $I$ within each sphere $= 3X/4\pi$ (§ 220). If $v$ = volume of all the spheres per unit volume of the mixture, the magnetic moment of the mixture per unit volume $= Iv = 3hv/4\pi$.

If $F$ be the average field in the mixture, the field within the spheres is zero and that around the spheres is $h$, hence

$$F = 0 \times v + h(1 - v) = h(1 - v).$$

The apparent susceptibility

$$\begin{aligned} k = \frac{I}{F} &= \frac{3hv}{4\pi} \Big/ h(1 - v) \\ &= 3v/4\pi(1 - v), \end{aligned}$$

and

$$\mu = 1 + 4\pi k = (1 + 2v)/(1 - v).$$

**221. Transversely Magnetised Cylinder.**—A very long cylinder is uniformly magnetised in a direction per-

pendicular to its axis. Find the field at any point inside or outside.

Let $a$ = radius. As in the case of the sphere (§ 111) replace the uniform magnetisation by a set of doublets of length $l$, pole strengths $\pm m$, uniformly distributed so that there are $n$ per unit volume. All the north poles collectively form a cylinder of uniform magnetic density $mn$.

By theory of attractions, this would produce a field at an external point $P$ (Fig. 116) $= 2\pi a^2 mn/NP$ along $NP$, and the south magnetism exerts a force $2\pi a^2 mn/SP$. These forces along $NP$ and $PS$ are in the ratio $SP : PN$; consequently the triangle of forces at $P$ is similar to $\angle SPN$, and the resultant $PT$ makes angle $SPT = \angle PNS$. Thus a circle described about $SPN$ touches $PT$. In limit, when $S$ and $N$ coincide, this circle touches the axis of magnetisation $OA$ at the centre $O$; hence the external lines of force are circles which would all touch one another at $O$; see Fig. 118.

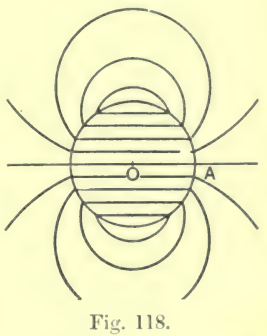

Fig. 118.

The magnitude of resultant : component along $NP = SP : SN$; therefore the field $= 2\pi a^2 mnl/SP.PN$; but $\pi mnl =$ magnetisation per unit volume $= I$. Hence the field at $P = 2a^2 I/OP^2$, and it makes with $OP$ an angle $OPT = 180° - POA$.

For a point $Q$ inside, the forces from $N$ and to $S$ are respectively $2\pi mn.NP$, $2\pi.mn.PS$; hence their resultant is parallel to $AO$ and $= 2\pi mnl = 2\pi I$. The induction $= 4\pi I - 2\pi I = 2\pi I$. Fig. 118 shows the lines of induction inside and out.

**Example.**—A cylinder of permeability $\mu$ is placed transversely in a field $h$. Find the magnetisation produced in it, and the induction.

If $I$ be the magnetisation, this causes an internal field $-2\pi I$, the sign showing its direction. The total internal field is therefore

$$H = h - 2\pi I.$$

But

$$I = kH = (\mu - 1)H/4\pi,$$
$$\therefore\ 4\pi I = (\mu - 1)H = (\mu - 1)h + (\mu - 1)2\pi I,$$
$$\therefore\ 2\pi I = (\mu - 1)h/(\mu + 1).$$

And

$$H = 4\pi\mu/(\mu - 1) = 2h/(\mu + 1),$$
$$\therefore\ B = \mu H = 2\mu h/(\mu + 1).$$

In the limit when $\mu$ is infinite, $B$ is only $2h$.

**222. Molecular Currents. Diamagnetism.**—Let a magnetic molecule be supposed to consist of a circuit carrying an electric current, and having inductance $L$, but no resistance at all. If there were any resistance, the current would perish, and magnetic properties would not be permanent. Let first the circuit be fixed in position, and an external magnetic field begin to act upon it. This would generate an induced current, so that the charge circulating during the change

$$= \text{Change of Flux/Resistance.}$$

But the resistance is zero, and the charge cannot be infinite, therefore the total change of flux must be zero. That is, the change $F$ produced by the external field is exactly equal and contrary to the change of flux produced by altering the current. The current therefore alters by $F/L$.

Now, by Lenz's law, the magnetic action of the extra current $F/L$ opposes the change of field which produced it. So a change of magnetic field $\delta H$ causes every molecule to alter its magnetic action by an amount proportional to $\delta H$ and opposing it. The medium, regarded as space containing a large number of such molecules, is therefore magnetised so that its intensity $I$ is proportional to $H$ and has the contrary sign. The susceptibility of the medium is therefore negative, and the medium behaves as a diamagnetic one.

This can be illustrated experimentally. A sphere of bismuth [diamagnetic] is repelled from the space between the pointed pole pieces of a powerful electromagnet. A sphere of copper in the same position would be repelled if

the poles were suddenly magnetised by a strong current; for the increasing field would induce a current in the copper, and repel this induced current circuit before it had time to decay.

**223. Molecular Currents. Paramagnetism.**—We next suppose that the current circuits are not fixed in position. The molecule is then equivalent magnetically to a small equivalent bar magnet set perpendicular to the plane of its resultant current circuit. In any unmagnetised medium, the magnetic molecules have all directions indiscriminately. If a magnetic field act, it tends to get the axis of every one of these equivalent magnets parallel to itself. The magnetic molecules are not supposed absolutely free; their deflections will be resisted, and each one will yield to the field by an amount which is more or less proportional to the field strength. Each unit volume of the medium will acquire a magnetic moment proportional to the total effect of the field in directing the equivalent magnets parallel to itself.

The magnetic moment acquired through the mobility of the molecules is therefore in the same direction as the magnetising field. So this cause produces paramagnetism.

Thus essentially diamagnetism is due to induced currents, and is proportional in amount to the coefficient of mutual induction of the molecular currents. Paramagnetism is proportional to the freedom with which magnetic molecules can be rotated. A vacuum has, probably, no magnetic molecules. Every other substance would have both the paramagnetic and the diamagnetic property; but in some substances the first would preponderate and in others the second. The diamagnetic property is always very weak. The paramagnetic is weak except in a few substances; we therefore infer that molecules are nearly always held with considerable rigidness.

This being so, it follows that the deflections of the molecular magnets are very small in most cases, even in the intensest fields, except within iron, nickel, etc. Small deflections are usually proportional to the deflecting force; hence $k$ is a constant for every one of the many sub-

stances (whether paramagnetic or not) whose susceptibility is small.

**224. Molecular Currents. Saturation. Coercive Power.**—In mechanical systems large deflections are usually not proportional to the deflecting force. We therefore expect $k$ not to be constant in iron. Besides, if deflections exceed certain limits, a system is generally so distorted that it tends to return to an altered equilibrium position when the deflecting force is removed. Often a limit exists beyond which no distorsion can pass. Thus, in the case we are considering, no magnetic field, however great, could do more than set all the magnetic axes of the molecules parallel to itself; and only an infinite field could do as much.

We should expect therefore that very magnetic substances would show the following properties:—

*Saturation.*—That there is a limiting intensity of magnetisation ($I$) which can only be approached but is essentially unattainable.

*Coercive Power.*—That if a large enough field be applied and then removed, the substance does not return to its original state but shows (more or less) residual magnetisation, of the same kind as was produced by the field but of smaller magnitude.

**225. Hysteresis. Irrationality.**—Another property that we might not, perhaps, have expected is that magnetic substances do not instantaneously take their final equilibrium state in any applied field. In a strong field the intensity increases gradually; when the field is removed it falls gradually; and if the field is first increased and then diminished the maximum intensity comes later than the maximum field. This property is called **Hysteresis.** It has mechanical analogies. A wire under great tension does not instantly take its full length; and if the tension be alternately increased and diminished, the maximum length is later than the maximum tension.

Even for moderate fields the magnetisation of iron is not strictly proportional to the magnetising force. This

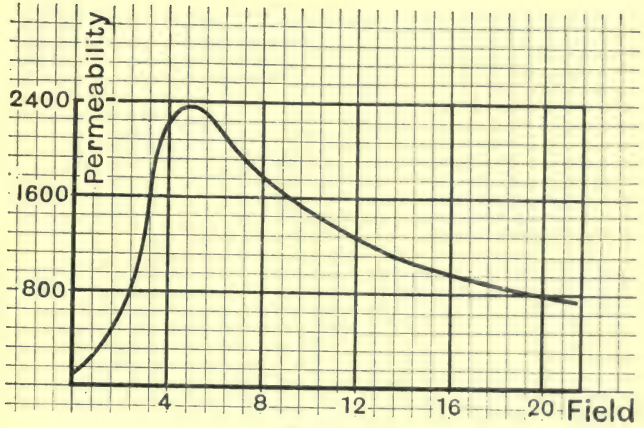


Fig. 119.

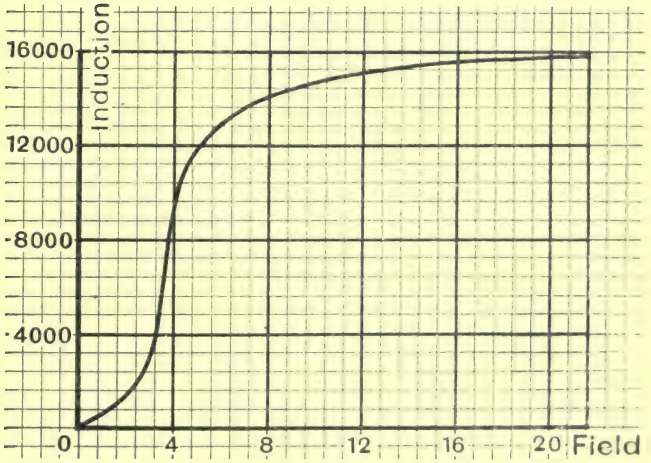


Fig. 120.

property may be called **Irrationality**; and is expressed mathematically by the fact that $k$ (and therefore $\mu$) is not constant. In Figs. 120, 119 typical curves are given showing how $B$ and $\mu$ alter with $H$. Of course we define $k$ and $\mu$ by the same equations as before, namely

$$I = kH, \quad B = \mu H, \quad \mu = 1 + 4\pi k.$$

As $I$ is finite when $H$ is infinite, $k$ or $I/H$ tends to limit zero, and $\mu$ tends to limit 1.

**226. Total Flux.**—Let $A$ be the cross section of any portion of a bar of iron or other magnetic substance. Let $H$ be the longitudinal field; then the induction $B = \mu H$. There are $B$ lines of induction per unit area, hence the total flux or total number $F$ of lines crossing the section $A = AB = \mu HA$. Let $x$ be any length of the bar, and $V$ the difference of magnetic potential at its ends. Then $V$ is the work done by a force $H$ acting for a distance $x$; or

$$V = Hx, \; H = V/x.$$

Hence the total number of lines crossing the section

$$= F = \mu A V/x.$$

Or

$$F/V = \mu A/x.$$

Compare this with the formula for flow of electricity, $C = V/R$ (§ 111), where $C$ is the current, $V$ the P.D., and $R$ the resistance. If $R_0$ be the specific resistance, and $x$ and $A$ the length and cross section,

$$R = R_0 x/A,$$
$$\therefore \; C/V = A/xR_0.$$

Therefore the permeability $\mu$ has a certain analogy with $1/R_0$, the specific conductivity. Also the resistance $R$ is analogous to $x/A\mu$; which quantity is called the **reluctance.** Of course $1/\mu$ can be called the **specific reluctance.**

In virtue of the above analogy, we can (as in §§ 109, 112) infer that if two reluctances be in series the resultant

reluctance is their sum; and if two be in parallel, of magnitudes $r_1$ and $r_2$, their resultant reluctance is

$$= r_1 r_2/(r_1 + r_2).$$

**227. Magnetic Circuit.**—We have seen that lines of magnetic induction do not start or terminate anywhere, and either go both ways to infinity, or form re-entrant curves. The lines due to a current circuit are re-entrant, and loop round the current.

In such cases any one bundle of lines which is followed completely round until it joins itself again is called a Magnetic Circuit. If there be an iron core looping round the current its great permeability causes as many lines as possible to crowd into it; and there is practically no great error in supposing the iron itself to form the circuit. But if a few lines stray into the air and rejoin the iron again the effect is that of an air reluctance in parallel with an iron reluctance much less than itself. Assuming that any lines which stray are negligibly few, take $F$ for the total number of lines in the iron.

For every section, $F = \mu A V/x$, or $Fx = \mu A V$, where $V$ is the work that would be done on a unit $N$. pole in travelling the short distance $x$. Adding up for the whole circuit, of length $l$ say,

$$Fl = \mu A \Sigma V.$$

Here $\Sigma V$ is the sum of the work done on a unit $N$. pole for displacements over all the short distances $x$ which collectively form the complete circuit $l$. Hence $V$ is the "Magnetomotive Force" as defined in § 169. It is the same for all circuits which loop through the current circuit, whether they be described in iron or air, and its value is $4\pi nC$, where $C$ is the current (in absolute units) and $n$ the number of turns. The unit of M.M.F. [Magnetomotive Force] is called a Gilbert,* but engineers often

* The Gauss is the unit of Magnetic Induction and also of Field, the Gilbert of M.M.F., the Oersted of Reluctance, the Weber of Total Flux. These names are not in common use, and seem rather confusing than helpful. We do not recommend anyone to remember them.

estimate it in "ampere-turns." If the current be 1 amp. and there be 1 turn, then $C = 1/10$ and $n = 1$, hence the "ampere turn" $= 4\pi/10$ Gilberts

$$= 1{\cdot}257 = 5/4 \text{ nearly.}$$

*Note.*—In the next examples we are taking $\mu = 200$, and therefore $k = (\mu - 1)/4\pi = 15{\cdot}8$. This is quite arbitrary, for Fig. 119 shows that $\mu$ can have all values from about 100 to 2400 for the same iron in different fields. Also the values of $\mu$ differ widely for different specimens of iron in the same field, and for the same iron at different temperatures. Apparently the highest permeability observed is about 12,000, for iron in a weak field, at about 770°, a temperature very little below the point at which magnetic properties vanish.

**Ex. 1.**—Find the flux around an iron ring of mean radius 8 cm. The iron of the ring is of circular section and of diameter 1 cm. The ring is evenly wound with 500 turns of wire, which carry a current of ·1 ampere. Take $\mu = 200$.

The advantage of such an iron ring is that there are no free poles, and therefore no demagnetising force. The M.M.F. $= 500 \times {\cdot}1$ $= 50$ ampere turns $= 20\pi$ absolute units (Gilberts). The area of the cross section is $\pi/4$, and the length of the iron circuit is $16\pi$, therefore the reluctance

$$= 16\pi \div \{200\pi/4\} = {\cdot}32 \text{ oersted,}$$

and the flux

$$= 20\pi/{\cdot}32 = 125\pi/2$$

$$= 196 \text{ Webers, or } 196 \text{ absolute units.}$$

**Ex. 2.**—If the same wire carrying the same current were closely wound on a wooden ring of the same dimensions, what would the flux be?

The only difference in the calculations is that $\mu = 1$ instead of $\mu = 200$. The reluctance is 64, and the flux $= 20\pi/64 = {\cdot}98$.

**Ex. 3.**—If the wire had not been wound uniformly, but the 500 turns confined to only a portion of the ring (say a quarter), what difference would this make to the iron and the wood?

Iron is so much more permeable than air that hardly any lines of induction stray out of the metal. The magnetic circuit is confined (practically) to the iron, and its reluctance, and therefore its flux, is practically unaltered. Wood is equivalent to air. The lines run within the wood in the wound portion, whose reluctance is a quarter of the whole, and therefore is about 1·256. But at the ends of the wound portion the lines diverge into the air, and the reluctance outside the windings is therefore much diminished. The total re-

luctance is therefore nearly 1/256, hence the total flux, which is the flux in the wound part of the wood, is about four times as great as before; and the flux in most of the unwound wood is much smaller, as only a few lines lie within the wood.

**Ex. 4.**—The above iron ring and wooden ring are evenly wound by 500 turns of wire as before, but the radius of the winding is double that of the section of the ring. What difference does this make?

The M.M.F. is the same as before and so are the reluctances of the rings. Hence the flux in the iron is unaltered and the flux in the wood is unaltered. There is an additional flux in the air space within the windings. The sectional area of this is three times that of the wood or iron, therefore the flux in this space is three times that in wood, or 2·94. The total flux within the winding is therefore about 199 for the iron, 3·9 for wood. It is unimportant with an iron core whether the winding be even and whether it is close. But with a wooden core these things are important.

**Ex. 5.**—The iron ring of Question 1 has a saw cut of width 1 mm. made across its section. What influence will this have on the flux?

The reluctance of the cut (length 1/10, area $\pi/4$, $\mu = 1$) is $4/10\pi = \cdot1273$. The reluctance of the metal removed is got by dividing by 200 and is ·0006; but the reluctance of the whole metal is ·32 and therefore that of the metal remaining is ·3194. Adding, the total reluctance is ·4467, and the flux $= 20\pi \cdot4467 = 140$ nearly.

**Ex. 6.**—What would be the flux if the width of the saw cut were 1/10 mm.?

The flux is 189. This differs from the flux 196 of a complete ring by over 3%, and shows that corrections for an imperfect join are very large indeed.

**Ex. 7.**—What would be the effect of a saw cut of width 1 mm. extending only half across the section?

The reluctance of a complete cut was found to be ·1273, and of the metal removed ·1273 200. We have half the cut and half the metal, in parallel. If $r$ be the reluctance of the metal portion, the air portion is $200\,r$, and the two in parallel have reluctance

$$r \times 200\,r/(r + 200\,r) = 200\,r/201 = \cdot1273 \cdot402 = \cdot0003.$$

The total reluctance is ·3197, and the flux $= 20\pi \cdot3197 = 196$, practically the same as if there were no cut.

**Ex. 8.**—If two magnetic circuits of reluctance $A$ and $B$ are surrounded by the same electric current, show that their joint reluctance is $AB/(A + B)$. Show that this is true whether the

circuits be para- or dia- or even ferromagnetic, provided they do not appreciably influence one another.

**Ex. 9.**—A current $c$ (absolute units) flows in a long straight conductor. Round it is wound symmetrically insulated iron wire, of radius $r$ and permeability $\mu$, in a helix of length $l$, number of turns $n$, and radius $a$. The terminals being joined, calculate the flux in the helix. Show that the result is independent of $l$ and $n$.

**228. Induced Currents.**—In a field containing substances other than air let there be two circuits, which we shall call the Primary and the Secondary, carrying currents $c_1$ and $c$ (as in § 200). Let first every substance present be simply paramagnetic or diamagnetic, that is, every $\mu$ and $k$ is a constant. The magnetising field which is the direct result of $c_1$ is proportional to $c_1$ and produces induced magnetisation proportional to $c_1$. This magnetisation causes fields proportional to itself (and therefore to $c_1$) and acts as a demagnetising force in paramagnetic bodies and a supermagnetising force in diamagnetic, but the consequent magnetisation produced is proportional to the causative magnetisation (because $k$ is constant) and therefore to $c_1$. The subsequent effects due to this consequent magnetisation can all be discussed in the same way. Hence everywhere the final field, induction, and magnetisation which are due to $c_1$ are proportional to $c_1$ and independent of the current $c$ in secondary. The flux through the circuit of $c$ due to $c_1$ is therefore $Mc_1$, where $M$ is a constant. All the mathematical consequences explained in §§ 200—210 follow, and $M$ can be called the coefficient of mutual induction, and defined in any of the three ways given in § 201.

**229. Influence of Ferromagnetic Substances.**—Things are less simple if ferromagnetic substances are present. The original magnetising force due to $c_1$ is still proportional to $c_1$. But the magnetisation directly produced by it is not proportional to it, but depends on $c_1$ in a more complicated manner and depends likewise on any other magnetisation previously existent in the iron, etc., whether permanent or due to some other current. The demagnetising force due to this magnetising is everywhere

proportional to it; but the consequent magnetisation due to this demagnetising force is no longer proportional to it but depends on the previous magnetisation. Continuing this argument we find that the final resultant flux due to $c_1$ which goes through $c$ is not only not a constant multiple $Mc_1$ of $c_1$, but is not even a function of $c_1$ only. And the mutual energy, instead of being $Mcc_1$ (§ 200), has to be written $W = \phi(c_1, c)$, where $\phi$ is a function which is not symmetrical in $c_1$ and $c$, is not even dependent only on $c_1$ and $c$, but is dependent on the previous history of the iron (or other ferromagnetic substance) in such a way that $\phi$ alters continually in the course of an experiment.

**230. Definition of Mutual Inductance.**—For convenience, one often speaks of the mutual inductance of a pair of coils (*e.g.* a "Transformer") which has an iron core. But the only sense which can be attached to the term is the following:—

The mutual inductance of two coils associated with ferromagnetic substances is the coefficient of mutual induction of two coils in air which would exhibit the same induction phenomena under the precise circumstances of the experiment considered.

Hence mutual inductions can be found or compared by any method previously described (§ 205 onward). Similar remarks apply to self-induction, which is only the mutual induction between the coil and itself. But every different method will give a different value to the induction coefficients measured, and the same method will give different values if currents of different magnitudes be employed. Scientifically, one ought never to give a precise value to any of the induction coefficients of coils with iron cores: and no surprise should be felt if two determinations give results which are not even of the same order of magnitude.

**Ex. 1.**—If the cores of Examples 1, 2, 5, 6, 7 (§ 227), be evenly wound with 10000 turns forming a secondary circuit, in addition to the primary of 500 turns. Find the coefficients of mutual induction, treating iron as a "simply paramagnetic" substance ($\mu = 200$).

If the primary current be ·1 ampere, it is $10^{-}$ absolute units.

The flux due to this current, in the first example, is 196 lines. But each line passes $10^4$ times through the secondary, so we must put $\delta f = 1{\cdot}96 \times 10^6$ when $\delta c = 10^{-2}$, therefore $M = 1{\cdot}96 \times 10^8$ absolute units.

To reduce to henries we divide by $10^9$; and the result is ·196. The result (on the false supposition made about iron) may be regarded as a constant for the coils. The answers for Examples 2, 3, 4, 5, 6, 7 are (in the same way) got by dividing the fluxes by 1000. The result ·00098 for Example 2 is accurate for all currents, since no iron was used.

**Ex. 2.**—If a primary have $n$ turns, and secondary $N$ turns, and the reluctance of the largest magnetic circuit which penetrates both coils be $R$, show that the coefficient of mutual induction is $4\pi Nn/R$.

The M.M.F. due to current $c$ for $n$ turns is $4\pi nc$. The flux in the circuit is $4\pi nc/R$. This penetrates $N$ times, therefore the flux in the secondary is $f = 4\pi Nnc/R$. But $M = f/c$, therefore

$$M = 4\pi Nn/R \text{ in absolute units.}$$

**Ex. 3.**—In a certain sample of iron $1/\mu = {\cdot}0025 + {\cdot}0001\ H$. If a ring have the same dimensions as before (radius of ring = 8, radius of section = $^1/_2$) and a primary and secondary of 500 and 10000 turns, calculate the apparent coefficient of mutual induction when the primary current is suddenly altered from 0 to ·01, ·1 and 1 ampere respectively.

Taking $c$ = ·01 amp. = ·001 absolute unit, the M.M.F. = $4\pi \times 500c = 2\pi$ and the circumference is $16\pi$, hence $H = 1/8$. Hence

$$1/\mu = {\cdot}0025 + 1/80000.$$

The reluctance would be 64 for a wooden core; its actual value is therefore

$$64\{{\cdot}0025 + 1/80000\} = {\cdot}16 + 8/10000 = {\cdot}1608.$$

Hence in absolute units

$$M = 4\pi \times 500 \times 10000/{\cdot}1608.$$

Reducing to henries by dividing by $10^9$ we have

$$M = {\cdot}38 \text{ henry.}$$

If $c$ = ·1 amp., the reluctance

$$R = 64\{{\cdot}0025 + 1/8000\} = {\cdot}168,$$

hence

$$M = {\cdot}37.$$

If $c$ = 1 amp.,

$$R = 64\{{\cdot}0025 + 1/800\} = {\cdot}24,$$

$$\therefore\ M = {\cdot}26$$

**Ex. 4.**—In the same example what would the coefficients of mutual induction become for the same currents if the primary and the secondary were interchanged?

For the same currents, the fields $H$ are 20 times as large as before. The reluctances work out to ·176, ·32, 1·76. Consequently the values of $M$ are ·36, ·196, ·036 henries.

**Ex. 5.**—If the permeability $\mu$ be given by an equation of the form $1/\mu = \alpha + \beta H$,* as in Examples 3 and 4, show that the total flux $f$ is a simple function of the primary current. Obtain expressions for $M$.

Consider the case of a core of length $l$, wound with $n$ and $N$ turns as before, and of radius $a$. Then the magnetising field is

$$H = 4\pi nc/l.$$

Therefore the total flux is $f = \mu H \,.\, \pi a^2 N$

$$= \frac{\pi a^2 N H}{\alpha + \beta H} = \frac{4\pi^2 a^2 N n c}{l\alpha + 4\pi n \beta c}.$$

It is therefore expressed as a function of $c$.

If $M$ be measured by a make and break experiment,

$$M = f/c = \frac{4\pi^2 a^2 N n}{l\alpha + 4\pi n \beta c}.$$

But if $M$ be measured for very small current changes,

$$M = \frac{df}{dc} = 4\pi n. \frac{df}{dx} = \frac{4\pi^2 a^2 N n l \alpha}{(l\alpha + 4\pi n \beta c)^2}$$

by methods of differential calculus.

It is easily seen that $M$ is smaller in the second case, and for large values of $c$ it can be very small indeed.

The formula $1/(\mu - 1) = \alpha + \beta H$ can be treated similarly.

**Ex. 6.**—The primary consists of $n_1$ turns of a considerable radius $a$; the secondary is $n_2$ turns wound in the same coil but insulated from the primary. Around this coil is wound insulated iron wire, binding the primary and secondary together and forming a re-entrant helix of length $2\pi a$ and area of cross section $\pi b^2$. If there are $N$ turns, and the ends of the iron wire are joined, calculate the mutual induction.

**Ex. 7.**—An iron tube, length $l$, external and internal radii $a \pm \frac{1}{2}b$, has the primary wire looped $n_1$ times through it, and the

* A more accurate formula is $1/(\mu - 1) = \alpha + \beta/H$. This gives $\mu = 1$ when $H$ is infinite, which agrees with § 225.

secondary wire $n_2$ times. Calculate the mutual induction. Find also the self-inductance of each wire.

**Ex. 8.**—If a self-inductance is to be made of a flat coil of $m$ turns of copper wire, bound into the appearance of an anchor ring by $n$ turns of insulated iron wire forming a reentrant helix with its ends connected, show that the self-inductance has the same value as if the iron wire formed $n$ turns of a flat coil and the copper made a helix of $m$ turns around it.

**Ex. 9.**—Find the self-inductance if the electrical circuit forms a coil of 100 turns of radius 5 cm., and the iron wire is 1000 turns of iron wire of radius ·5 cm. wound tightly as a helix round the strands. The radius of the iron wire is ·02 cm., and permeability 200.

**231. Energy of Magnetic Field, if $\mu$ be constant.**—Let a uniformly wound ring, as described in § 227, Ex. 2, have mean circumference $l$, area of cross section $A$, and number of turns $n$. If the current is $c$, the field inside will be

$$H = 4\pi nc/l.$$

The Induction is

$$B = 4\pi nc\mu/l,$$

therefore the total flux in core

$$= 4\pi nc\mu A/l.$$

This passes $n$ times through the circuit, consequently the total flux through circuit is

$$F = 4\pi n^2 c\mu A/l,$$

therefore the coefficient of self-induction is

$$L = F/c = 4\pi n^2 \mu A/l.$$

The total electromagnetic energy (§§ 203, 228) is, if $\mu$ be constant,

$$W = \tfrac{1}{2}Lc^2 = 2\pi n^2 \mu A c^2/l.$$

Consequently the energy per unit volume is

$$\omega = W/lA = 2\pi n^2 \mu c^2/l^2.$$

Comparing with the values of $H$ and $B$ above, we find that the energy per unit volume is

$$HB/8\pi.$$

In air, $B = H$, so the result is $H^2/8\pi$.

**232. Energy of a Ferromagnetic System.**—When $H$ is the field, let $B$ and $I$ be the induction and intensity. They are supposed all to be in the same direction; or, if not, we use these letters to denote their components in the common direction $OX$ say. Consider an element of a shell of area $A$, perpendicular to the direction of $I$, and thickness $l$; so that the magnetic moment is $lAI$, and pole strength $AI$ (cf. § 211). If $I$ alters by $\delta I$, this is equivalent to a change $A\delta I$ of pole strength; and therefore we can regard $A\delta I$ units of $N$. magnetism to be moved a distance $l$ from the $S$. border of the shell to the $N$. As $H$ is the field, the force $HA\delta I$ acts through a distance $l$, and the work done is

$$= HAl\delta I.$$

Since $Al$ is the volume, the work done per unit volume is

$$\delta W = H\delta I.$$

Consequently the whole work expended in magnetising a system is

$$W = \int H\delta I,$$

the limits of integration representing the initial and final states of the system.

This can be put in another form. Since

$$B = H + 4\pi I, \quad I = (B - H)\,4\pi,$$
$$\therefore\ W = \int Hd(B - H)\,4\pi$$
$$= \int HdB\,4\pi - H^2/8\pi.$$

Now $H^2/8\pi$ is the energy that would exist in an air space, which is entirely unmagnetised. Hence the whole energy can be taken

$$= W + H^2/8\pi$$
$$= \frac{1}{4\pi}\int HdB.$$

If we put $B = \mu H$, this integral becomes (if $\mu$ is constant)

$$= \frac{\mu}{4\pi} \int H dH = \mu H^2/8\pi = HB/8\pi, \text{ as in 227.}$$

**233. Graphical Methods.**—Construct a graph in which abscissae represent values of $H$ and ordinates values of $I$. The point $P$ denotes a state in which the field $H = OM = NP$ and the intensity $I = MP = ON$.

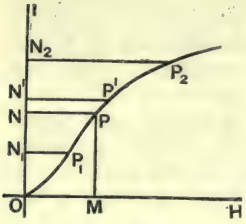

Fig. 121.

If the state alters to that represented by $P'$, supposed very near $P$, $NN' = \delta I$; hence $H\delta I = NP \times NN'$, which, in the limit when $\delta I$ is indefinitely small, can be regarded as = area $NPP'N'$.

Adding for a continuous sequence of changes from state $P$ to $P_2$, we find that the work done in magnetising is represented by the whole area $N_1P_1P_2N_2$ enclosed between the straight lines $P_1N_1$, $N_1N_2$, $N_2P_2$, and the graph $P_1P_2$.

If a graph were drawn in which the abscissae represented $H$ and the ordinates $B$, an area similarly described would represent

$$\int H dB$$

which $= 4\pi \times$ charge in the energy given to the magnetised body.

The energy given exceeds the work done in magnetising by the quantity $H^2/8\pi$ (taken between proper limits), which represents the energy which would have been given to a body which cannot be magnetised.

**234. Cyclic Processes.**—Let a sample of iron be subjected to a field which gradually increases from $-H_1$ to $+H_1$, then diminishes from $H_1$ to $-H_1$, then increases

again as before, and so on repeatedly. After a while, the intensity $I$ will alter cyclically, the curve described being similar to that shown. In the change $Q_2P_1$, the work done will = area $Q_2P_1N_1$. In the change $P_1Q_1$ (as the field is diminished) the work done is negative and = — area $Q_1P_1N_1$; so the algebraic sum for half the cycle is $Q_2P_1Q_1$. Similarly the total work provided in the other half cycle is $Q_1P_2Q_2$. The work provided in each complete cycle is the area of the closed curve $P_1Q_1R_1P_2Q_2R_2P_1$.

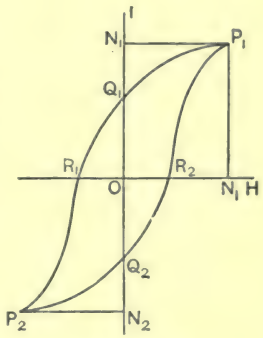

Fig. 122.

Everything has been restored to its original state, consequently no mechanical or electric energy remains, and the above work must be entirely expended in producing heat.

It may be remarked that the area of the closed curve is expressed analytically by $\int H dI$ taken all round; and this is equal to $I\dot{H} - \int \dot{I} dH$, or $- \int I dH$ simply. For $IH$ has the same initial and final value when we consider a complete cycle.

**235. Coercive Power. Demagnetising Field. Hysteresis.**—If the iron has been magnetised to state $P_1$, and $H$ is then made zero, the path $P_1Q_1$ is described and the iron is left permanently magnetised to an intensity measured by $OQ_1$. This is called its Coercive Power.

If the field is reversed to such a magnitude as just to destroy the magnetisation, this field is called the Demagnetising Field. The magnetisation is zero at $R_1$, consequently $OR_1$ measures the demagnetising force or field.

The general property of which these are effects is called Hysteresis (§ 225). It can be measured in practice by the area of the closed curve $P_1P_2$, that is, by the heat produced per cycle, when the cycle is defined by a given maximum field $ON_1$.

**236. Transformers and Alternate E.M.F.**—A transformer consists of a primary and secondary coil wound on a soft iron core. An alternating E.M.F. is supplied to the primary; and the induced secondary current is utilised for some purpose.

Let $c$, $C$ be the currents in primary and secondary, $r$ and $R$ resistances, $l$ and $L$ self-inductions. Let $M$ be the mutual induction, then $M^2$ is nearly $= Ll$ in most cases. We are supposing that $L$, $l$, $M$ can be treated as constants. Let the E.M.F. supplied to primary be $v$; as it is alternating we may put $v = v_0 \sin nt$.

Then the equations for the two coils are

$$v = l\dot{c} + M\dot{C} + rc, \tag{1}$$

$$o = M\dot{c} + L\dot{C} + RC. \tag{2}$$

We assume $C$ is periodic with same period as $e$,

$$\therefore\ C = C_0 \sin \overline{nt + \alpha}\ ;$$

and

$$\dot{C} = nC_0 \cos \overline{nt + \alpha}.$$

By (2),

$$\dot{c} = -(Ln \cos \overline{nt + \alpha} + R \sin \overline{nt + \alpha})\, C_0/M,$$

$$\therefore\ c = -(Ln \sin \overline{nt + \alpha} - R \cos \overline{nt + \alpha})\, C_0/Mn.$$

Substitute in (1)

$$v = -(Ln \cos \overline{nt + \alpha} + R \sin \overline{nt + \alpha})lC_0/M$$
$$+ MnC_0 \cos \overline{nt + \alpha} - (Ln \sin \overline{nt + \alpha} - R \cos \overline{nt + \alpha})\, rC_0/Mn.$$

But

$$v = v_0 \sin nt = v_0 \{\sin \overline{nt + \alpha} \cos \alpha - \cos \overline{nt + \alpha} \sin \alpha\}.$$

Hence, equating coefficients of $\cos \overline{nt + \alpha}$ and $\sin \overline{nt + \alpha}$,

$$nLlC_0/M - nMC_0 - RrC_0/Mn = v_0 \sin \alpha \tag{3}$$

$$- RlC_0/M - rLC_0/M = v_0 \cos \alpha. \tag{4}$$

These give $C_0$ and $\alpha$.

Let the secondary have $k$ times as many turns as the primary, then the flux through the secondary can be taken as $k$ times that in primary, for the two coils have a com-

mon core. Hence the rate of charge of flux per unit charge of $c$ is $k$ times as great in secondary as in primary, therefore $M = kl$.

Similarly $L = kM$; therefore $Ll = M^2$.

Using these relations, (3) and (4) give

$$Rr C_0/Mn = -r_0 \sin \alpha. \tag{5}$$

$$(R/k + rk)C_0 = -r_0 \cos \alpha. \tag{6}$$

$$\therefore\ C_0 = -r_0/\sqrt{\{R^2r^2/M^2n^2 + (R/k + rk)^2\}}$$

and

$$\cot \alpha = Mn\{1/rk + k/R\}.$$

**237. Transformers and Constant E.M.F.**—The equations are (1) and (2) as above, but $v$ is now a constant. Assuming $M/l = L/M = k$, we have

$$v = l(\dot{c} + k\dot{C}) + rc$$

$$o = lk(\dot{c} + k\dot{C}) + RC.$$

$$\therefore\ v = rc - RC/k.$$

Since initially $c = o$ and finally $c = v/r$, we assume

$$c = (1 - e^{-\lambda t})v/r,$$

$$\therefore\ v - rc = ve^{-\lambda t}.$$

Hence

$$RC/k = -ve^{-\lambda t},$$

and

$$ve^{-\lambda t} = l(\dot{c} + k\dot{C}) = l\lambda\{ve^{-\lambda t}/r + kve^{-\lambda t}/R\},$$

$$\therefore\ \lambda = 1/(l/r + kl/R)$$

and

$$-RC = kve^{-t(l/r + kl/R)}.$$

In an ordinary Ruhmkorff, $R$ is so large that it can be taken infinite, and $Rk$ is also large, but $C$ is indefinitely small and $RC$ is finite. Also $-RC$ is the E.M.F. induced at make. Its value is therefore $kv$.

At break, we have a current $c_0 = v/r$ already established; and the "break" only means replacing $r$ very suddenly by a much larger resistance $r'$. Hence $c = c_0$ initially, and $= v/r' = c_0 r/r'$ finally; so we assume

$$c = c_0\{r + (r' - r)e^{-\lambda' t}\}/r'.$$

As before,

$$v = l(\dot{c} + k\dot{C}) + r'c$$

and

$$o = lk(\dot{c} + k\dot{C}) + RC.$$

$$\therefore\ RC = k(r'c - v) = k(r'c - rc_0) = kc_0(r' - r)e^{-\lambda t}.$$

And this

$$= -lk(\dot{c} + k\dot{C})$$

$$= \lambda lkc_0 e^{-\lambda t}\{(r' - r)/r' + k(r' - r)/R\}.$$

$$\therefore\ = 1/(l/r' + kl/R).$$

The initial value of $RC$ is $kc_0(r' - r) = kv(r' - r)/r$, and is therefore very much larger than at make, since $r'$ is always great.

**238. Dynamos.**—The motion of a moving coil produces a periodic change of flux through it. We assume this change harmonic, and write $F = A \sin nt$. The resistance and inductance of the circuit are $L$ and $R$; hence the equation for the current is

$$\dot{F} = LC + RC.$$

Assume

$$C = C_0 \sin\ (nt + \alpha).$$

Then

$$An \cos nt = C_0\{Ln \cos \overline{nt + \alpha} + R \sin \overline{nt + \alpha}\}.$$

But

$$An \cos nt = An\{\cos \overline{nt + \alpha} \cos \alpha + \sin \overline{nt + \alpha} \sin \alpha\},$$

$$\therefore\ C_0 Ln = An \cos \alpha,$$

$$C_0 R = An \sin \alpha,$$

$$\therefore\ \tan \alpha = R/Ln,$$

$$C_0 = An/\sqrt{R^2 + L^2n^2}.$$

The zero current is when $nt + \alpha = o, \pi, 2\pi$, etc.; and the zero E.M.F. when

$$F = o,\ nt = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc.}$$

The zero current is later in phase than the zero E.M.F. by $\pi/2 - \alpha, = \tan^{-1}(Ln/R)$

The zero E.M.F. is later with an iron core than with a simply paramagnetic core owing to hysteresis. Consequently the zero current is later than we should expect, owing to two causes, namely hysteresis and self-induction.

If the dynamo be simply connected to the external circuit, we have an alternating current given by the above equations.

If a direct current is wanted, sliding contacts (brushes) are arranged to reverse the connections, and therefore alter the sign of $C$ in the external circuit, at the instants when $C$ vanishes. The brushes should be adjustable, because $C$ is made later in phase by hysteresis and self-induction as before. The correction for self-induction is $\tan^{-1}(Ln/R)$, and obviously increases with $n$. The correction for hysteresis does so also. The angle through which the brushes are moved is called the Angle of Lead.

# CHAPTER XIII.

## THERMOELECTRICITY.

**239. Peltier Effect.**—If a current flows across a junction of two metals $A$ and $B$, heat is absorbed or given out. This heat is proportional to the first power of the current $c$ (thereby differing from the Joule Effect, § 107, which is proportional to $c^2$) and to the time $t$. Measuring everything in absolute units we may say that the heat absorbed (in ergs) when a current $c$ flows for time $t$ from $A$ to $B$

$$= H = q\,c\,t.$$

If heat be given out, the multiplier $q$ is negative. Its value depends on the two metals and on the absolute temperature $\theta$. It is usually positive for a range of values of $\theta$ and negative for another range, vanishing for a temperature ($\Theta$) called the Neutral Point for the metals.

If the current be reversed, flowing from $B$ to $A$, the sign of the effect changes. Hence $q =$ the energy reversibly supplied per unit current per second; it may therefore be called the E.M.F. at the junction $AB$ (§ 102).

**240. Second Law of Thermodynamics.**—Consider a circuit of the two metals $A$ and $B$, with junctions at temperatures $\theta$ and $\theta'$ ($\theta > \theta'$). Let the current flow from $A$ to $B$ at $\theta$, so that the warmer junction absorbs $H = qct$. The current flows from $B$ to $A$ at $\theta'$, and the colder junction gives out $q'ct$.

We shall assume that all other heat reversibly taken in or given out is negligible in comparison. As we shall see

later, this is true if $\theta - \theta'$ be small enough. Hence the whole energy reversibly supplied is

$$H - H' = (q - q')ct$$

so that there exists a resultant E.M.F. of value $q - q'$ in the circuit. The existence of such an E.M.F. was proved by Seebeck.

The Thermal Efficiency is defined, in thermodynamics, as $= (H - H')/H$; and is equal to $(\theta - \theta')/\theta$ if the temperatures are reckoned on the absolute scale.

Hence

$$\begin{aligned}(\theta - \theta')/\theta &= (H - H')/H \\ &= (q - q')/q.\end{aligned}$$

Or, taking the limit when $\theta - \theta'$ (and therefore $q - q'$) is small,

$$\frac{q}{\theta} = \frac{dq}{d\theta}.$$

**241. Thermoelectric Height.**—This quantity, $\frac{q}{\theta}$ or $\frac{dq}{d\theta}$, will be called the Thermoelectric Height of $A$ over $B$, and will be denoted by $h$. It has two physical definitions:—

1. Since $h = q/\theta = H/\theta ct$, the *Thermoelectric Height* of $A$ over $B$ = the quotient of the heat reversibly absorbed by unit current flowing from $A$ to $B$ for one second by the Thermodynamic Absolute Temperature.

2. Since $h = dq/d\theta =$ limit of $(q - q')/(\theta - \theta')$, and $q - q'$ is the E.M.F. of the circuit, the *Thermoelectric Height* of $A$ over $B$ = the E.M.F. at mean temperature $\theta$, per unit temperature difference (1° absolute), if the resulting current flows from $A$ to $B$ at the hotter junction.

It is found by experiment that the effects described are the same whether the metals $A$ and $B$ be simply pressed together or united by solder or a thin layer of some other metal at the junction. If this third metal be $L$, and current $c$ pass from $A$ through $L$ to $B$, let $h_1$ and $h_2$ be the thermoelectric heights of $A$ and $B$ over $L$. Then the heat absorbed is $h_1 ct$ at junction $AL$, and $-h_2 ct$ at $LB$; but the total is $hct$. Hence

$$h = h_1 - h_2.$$

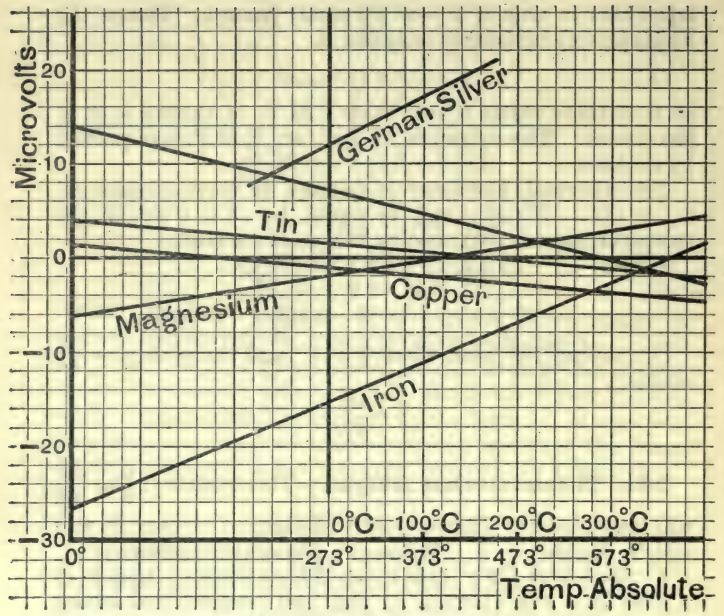


Fig. 123.

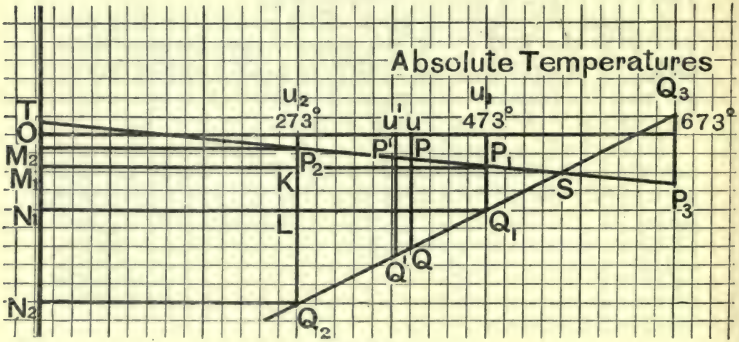


Fig. 124.

For convenience, all tabulated thermoelectric heights are reckoned from lead, taken as zero. The reason of the choice is given later (§ 245). We extract from Lupton's Numerical Tables the following formulae [$t$ = the centigrade temperature] for the thermoelectric height in microvolts of certain metals. The values given represent an ideal for a certain standard of purity and texture.

| | | | |
|---|---|---|---|
| *Iron* | $-17{\cdot}34 + {\cdot}0487t$ | *Copper* | $-1{\cdot}36 - {\cdot}0095t$ |
| *German Silver* | $12{\cdot}07 + {\cdot}0512t$ | *Tin* | $+{\cdot}43 - {\cdot}0055t$ |
| *Magnesium* | $-2{\cdot}24 + {\cdot}0095t$ | *Aluminium* | $+{\cdot}77 - {\cdot}0039t$ |

These data are true from $-18°$ to $+416°$ except for German Silver.

The graphs of $h$ against $\theta$ are obviously straight lines as in Fig. 123.

Some text-books draw the figures as here, and others draw them with the sign of $h$ everywhere reversed. In thermodynamics the indicator diagrams are described clockwise when heat is taken in at the higher temperature. Hence the ordinate for the higher temperature should be described downwards in that case.

**242. E.M.F. of a Thermal Circuit.**—Let the temperatures of junctions be $\theta_1$ and $\theta_2$, $\theta_1$ being larger. The Fig. 124 is drawn for copper and iron, the temperatures being 473° and 273° absolute ; 200° and 0° centigrade.

Divide the considerable difference of temperature $\theta_1 - \theta_2$ into small portions of which one is $\theta - \theta'$, where $\theta$ and $\theta'$ are represented by $OU$ and $OU'$ in Fig. 124. The thermoelectric heights of the first metal over the second are $PQ$ and $P'Q'$ ; they are ultimately equal if $\theta - \theta'$ be small enough and therefore the E.M.F. of a circuit with junctions at $\theta$ and $\theta'$ would be

$$= (\theta - \theta')\, PQ = U'U.PQ$$
$$= \text{area } PQQ'P', \text{ ultimately.}$$

We assume that the E.M.F. for the finite range of temperatures $\theta_2$ to $\theta_1$ is equal to the algebraic sum of the E.M.F.'s for the ultimately vanishing portions $U'U$ into which the range is divided. But the sum of such areas as $PQQ'P'$ is the whole area $P_2P_1Q_1Q_2$.

Hence the E.M.F. between two given temperatures is represented by the area bounded by the two ordinates $P_1Q_1$ and $P_2Q_2$ corresponding to the temperatures, and the thermoelectric diagrams $P_2P_1$ and $Q_1Q_2$ for the metals concerned.

**243. Direction of the E.M.F.**—If in Fig. 124 the temperatures be 273° and 473° absolute (or 0° and 200° C.), the E.M.F. is represented by area $P_1Q_1Q_2P_2$ and the current flows from copper to iron ($P_1$ to $Q_1$) at hot junction, so that the diagram is described clockwise.

If the temperatures had been 0° and 400° C., 273° and 673° absolute, the figure $P_2P_3Q_3Q_2$ crosses itself at $S$. If we start the cycle at $S$, we can describe it in the order $SP_3Q_3SQ_2P_2S$ ; and it clearly is the algebraic sum of two triangles, $SP_3Q_3$, described counterclockwise and therefore taken negative, and $SQ_2P_2$, clockwise and positive ; the sum of these triangles happens to be positive because the positive triangle is the larger. Hence if the junctions were heated to these temperatures a current would actually be produced in the clockwise direction $P_2P_3Q_3Q_2$; and it is obvious that heat would be given out at both junctions, and can be taken in only along the wires (see §§ 244, 245).

If the temperatures were 200° and 400° C. (473° and 673°) the total area $P_1P_3Q_3Q_1$ will be negative, because the negative triangle is numerically larger than the positive one. Hence the current would not flow in this direction but the contrary one, $P_3P_1Q_1Q_3$, and heat would be taken in at both junctions.

With temperatures 300° and 400° C. (573° and 673°) the current would describe the clockwise circuit $P_3P_4Q_4Q_3$.

Similar statements apply to other metals at other temperatures, whether the thermoelectric lines are straight or not.

**Ex. 1.**—From the tables given, find E.M.F. of a copper iron circuit for temperatures 0° and 100° C.

The height of the copper over iron is

$$h = (1{\cdot}36 - {\cdot}0095t) - (17{\cdot}34 + {\cdot}0487t) = 15{\cdot}98 - {\cdot}0582t.$$

At $t = 0$ and $t = 100$ the heights are 15·98 and 10·16. The area

of the trapezium = mean of parallel sides × distance between
$= \frac{1}{2}(15{\cdot}98 + 10{\cdot}16) \times 100 = 1307$ microvolts $= 1{\cdot}307$ millivolts.

Otherwise thus: The height at the average temperature

$$= 15{\cdot}98 - {\cdot}0582 \times 50 = 13{\cdot}07,$$

hence the area

$$= 13{\cdot}07 \times 100 = 1307 \text{ microvolts.}$$

**Ex. 2.**—At what temperature is the height of copper over iron = zero?

$$15{\cdot}98 = {\cdot}0582t, \therefore t = 278°.$$

**Ex. 3.**—What would be the E.M.F. for temperatures 0° and 556°?

This is found to be zero.

**Ex. 4.**—One copper iron junction is kept at 25°; the other is gradually raised from 0° to 600°. State generally what currents would be observed.

A current is observed at 0°. This diminishes till 25° is reached, then vanishes. It changes sign and increases till 278° is reached; then it diminishes as the temperature rises, until it vanishes at 521°. It then takes the original signs and increases to 600°.

**Ex. 5.**—What is the neutral point for iron and German silver?

**Ex. 6.**—Find E.M.F. of a copper tin circuit, from 100° to 200° C.

**Ex. 7.**—Assuming the diagrams to be straight lines, find E.M.F. of a copper German silver circuit between zero centigrade and zero absolute.

**244. Thomson Effect.**—Imagine unit current to flow round the circuit $P_1Q_1Q_2P_2$ for unit time. We have seen that the heat taken in at the hot junction

$$= H_1 = \text{thermoelectric height and temperature}$$
$$= P_1Q_1 \times OU_1 = \text{area } P_1Q_1N_1M_1.$$

The heat given out at the cold junction

$$= H_2 = P_2Q_2 \times OU_2 = \text{area } P_2Q_2N_2M_2.$$

And the heat converted into work is the E.M.F.

$$= W = P_1Q_1Q_2P_2.$$

We might have expected that $W = H_1 - H_2$: but obviously this is not the case in general. By figure,

$$H_1 - W - H_2 = P_1Q_1N_1M_1 - P_1Q_1Q_2P_2 - P_2Q_2N_2M_2$$
$$= P_1Q_1N_1M_1 - P_1Q_1Q_2N_2M_2P_1$$
$$= -P_1M_1M_2P_2 - Q_1Q_2N_2N_1.$$

Hence the heat utilised and given out ($W + H_2$) exceeds the heat taken in ($H_1$); therefore other heat must be taken in at some other place. But if it be not absorbed at the junctions, it must be absorbed in one wire or the other, or in both.

If the current were reversed, this absorption would be changed into production of heat.

The reversible production or disappearance of heat in a wire, whose ends are not at the same temperature, when it is traversed by an electric current, was deduced by Sir William Thomson from thermodynamic principles, and afterwards verified by experiment.

**245. Graphic Representation of Peltier and Thomson Effects.**—If the first metal be lead, its thermal diagram is the axis of coordinates $OU_2U_1$. Hence the heat absorbed in the wires of a lead-copper circuit (per unit current per second)=area $Q_1Q_2N_2N_1$. By experiment, no reversible production or emission of heat has been detected in a lead wire. This is the reason why the lead thermoelectric diagram is chosen as the coordinate axis. Then the area $Q_1Q_2N_2N_1$ = heat absorbed per second in a copper wire when a unit current flows from the hotter to the colder end.

Returning to the iron-copper circuit, we see that $P_1M_1M_2P_2$ = the heat absorbed (per unit current per second) when a current flows in an iron wire from the colder to the hotter end. Heat is given out if a current flows in iron from the hotter to the colder end.

We have now interpreted all the areas in the figure. $P_1Q_1N_1M_1$ is the Peltier Effect at the hot junction (per unit current per second), *i.e.* the heat taken in. $P_2Q_2N_2M_2$ is the Peltier Effect at the second junction. $P_1M_1M_2P_2$ and $Q_1N_1N_2Q_2$ are the Thomson effects in the two wires. Each of these areas represents heat taken in when it lies on the right hand side of the current, as in the portions $P_2P_1$, $P_1Q_1$, $Q_1Q_2$ of the circuit described clockwise. $P_1Q_1Q_2P_2$ is the total heat taken in (per unit current per second) and is therefore the E.M.F. due to the temperature difference. It is positive if the area be surrounded in a clockwise direction.

**246. Thermoelectric Formulae: E.M.F.**—Take the usual case in which the diagrams are straight lines. Then the thermoelectric heights $h_1$ and $h_2$ of the metals over lead are given by

$$h_1 = a_1 + b_1 t \ , \ h_2 = a_2 + b_2 t$$
$$\therefore \ h = h_1 - h_2 = a_1 - a_2 + (b_1 - b_2)t.$$

If we have two temperatures $t$ and $t'$ (centigrade), and $T$ be the neutral point, then $h = o$ at temperature $T$.

$$\therefore \ a_1 - a_2 = -(b_1 - b_2)T$$
$$\therefore \ h = (b_2 - b_1)(T - t).$$

The E.M.F. is represented by a trapesium whose parallel sides are $h$ and $h'$ corresponding to temperatures $t$ and $t'$. Then

$$h' = (b_2 - b_1)(T - t')$$

and E.M.F.

$$= \tfrac{1}{2}(h + h')(t - t')$$
$$= (b_2 - b_1)\{T - \tfrac{1}{2}(t + t')\}\{t - t'\}.$$

Hence the E.M.F.

= Difference of temperatures × difference between the mean temperature and the temperature $T$ of the neutral point × a constant for the metals which is the difference of their gradients.

All these formulae involve the coefficients $b$, $b'$. The $a$'s do not appear except in calculating the neutral point.

**247. Formulae for Peltier and Thomson Effects.**—The Peltier effect at $t$

$$= h(t + 273)$$
$$= (b_2 - b_1)(t + 273)(T - t),$$

= absolute temperature × difference of temperature and neutral point × a constant which is the difference of gradients for the metals considered.

This gives the heat per unit current (absolute) in ergs.

The Thomson effect in the first metal say iron, Fig. 124

$$= \text{area } P_1 M_1 M_2 P_2$$
$$= \tfrac{1}{2} M_1 M_2 (M_1 P_1 + M_2 P_2).$$

Now $M_1 M_2$ = difference of heights at the temperatures $t_1$ and $t_2$,

$$= a + bt_2 - (a + bt_1) = b_0 (t_1 - t_2)$$

and

$$M_1P_1 + M_2P_2 = 273 + \tfrac{1}{2}(t_1 + t_2).$$

Also, for iron, heat is absorbed as the current runs from colder to hotter.

Reversing the sign, so as to get the standard case where heat is absorbed as the current flows from the hotter end, we find the Thomson effect $= b\,(t_1 - t_2)\,(273 + \frac{1}{2}t_1 + t_2)$ = Gradient of the metal × Difference of temperatures × mean absolute temperature.

**Ex. 1.**—Wires of three metals are joined in series, and the three junctions are at different temperatures. Show how to find the E.M.F.

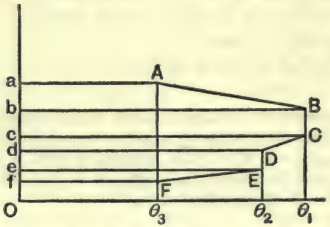


Fig. 125.

Let $AB$, $CD$, $EF$ be the thermal diagrams of the metals; and $BC$, $DE$, $FA$ the ordinates corresponding to absolute temperatures, $\theta_1$ $\theta_2$ $\theta_3$, as in Fig. 125. Imagine a unit charge to flow round the circuit. The Peltier effect at $\theta_1$ is that heat is taken in; and the quantity of heat is $BC$. $\theta_1$ = + area $BCcb$. The Thomson effect (heat taken in per unit charge) in metal $CD$ with junctions at $\theta_1$ and $\theta_2$ is = + area $CDdc$. Similarly the other Peltier and Thomson effects are + $DEed$, + $EFfa$, − $FAaf$, + $ABba$. The algebraic sum is the total heat provided per unit current, and is evidently the area of the hexagon $ABCDEF$. This energy is provided reversibly and is therefore equal to the E.M.F.

**Ex. 2.**—A copper wire and a German silver wire are connected to a galvanometer at the temperature of the room (16° C.). The other ends are soldered together and put in boiling water. If the total resistance be 1 ohm find the current in the galvanometer, the heat absorbed per second at the hot junction, and the heat absorbed per second in each wire. Compare the total heat taken in reversibly (algebraic sum of Peltier and Thomson effects) with the heat given out irreversibly (Joule effect). Use the data in § 241.

The equation to find the neutral point of copper and German silver is

$$-1{\cdot}36 - {\cdot}0095T = 12{\cdot}07 + {\cdot}0512T,\ T = -221.$$

The two wires are connected to one another directly at 100°, and, through the galvanometer, at 16°. An intermediate metal makes no difference if there is no temperature change. So the E.M.F. is that of a circuit with terminals at 100° and 16°. The difference is 84°; the difference between mean temperature and neutral point is 58 + 221 = 279. The E.M.F.

$$= (\cdot0512 + \cdot0095) \times 84 \times 279 = 1400 \text{ microvolts nearly.}$$

The resistance is 1 ohm, therefore the current is 1·4 milliamperes. In the complete diagram the current flows clockwise. This shows that it goes from German silver to copper at 100° C.

The Peltier heat absorbed at 100° per ampere second is

$$373 \times (100 + 221) \times (\cdot0512 + \cdot0095)10^{-6} = \cdot00727 \text{ joule.}$$

To get the actual heat absorbed per second, multiply by ·0014, the current in amperes. We get $1\cdot02 \times 10^{-5}$ joules = 102 ergs. Similarly the heat given out at the two cold junctions with the galvanometer = 58 ergs.

In the German silver wire, the temperature difference is 84° and the mean absolute temperature is 273° + 58° = 331°. The heat given out per ampere second = $\cdot0512 \times 84 \times 331 \times 10^{-6}$ joules $= 1\cdot42 \times 10^{-3}$; and the heat per ·0014 amp. = $2 \times 10^{-6}$ joules = 20 ergs. In the copper wire the heat given out is 4 ergs.

As no other work is done in the circuit except the irreversible production of heat, the total heat absorbed per second = the joule heat produced. Of course this heat = $RC^2 = 1 \times (\cdot0014)^2 = 2 \times 10^{-6}$ joules = 20 ergs, which = 102 − 58 − 20 − 4.

**Ex. 3.**—Find the two Peltier effects and the two Thomson effects per unit current for a copper-iron circuit, (1) with terminals at 0° and 100° C., (2) with terminals at 200° and 300° C.

**Ex. 4.**—A copper and an iron wire each 5 cm. long are joined together to form a rectangle 4 by 1 cm. Their joint resistance is ·012 ohm. If the rectangle is at a temperature near 15°, but the difference of temperatures of junctions is ·1°, calculate the magnetic moment of the rectangle.

**Ex. 5.**—A ring of 1 cm. radius is half of copper and half of German silver. Its resistance is ·003. Calculate the magnetic field produced at its centre when there is a difference of temperature of 1° between its terminals, at about the temperature of the room.

**Ex. 6.**—If the E.M.F. of a bismuth-platinum circuit with the terminals at 0° and 100° be given as 6500 microvolts, the current flowing from the bismuth at the hotter junction, show that the thermoelectric height of bismuth over platinum at 50° is 65. If the E.M.F. between − 190° and 0° be 12300, determine another

thermoelectric height, and express the height for bismuth by a graph and formula as in § 241.

**Ex. 7.**—The E.M.F. of a platinum and a platinum-iridium circuit, with one junction at zero centigrade, is given as 737 and 1571 microvolts respectively when the other junction is at 500° and 1000°. Deduce a formula (linear) for the relative thermoelectric height, and a formula for the E.M.F. between 0° and t° C.

**248. Abnormal Metals.**—The graph for nickel is shown in the diagram. It has three nearly straight portions, joined by short curves. Other magnetic metals have the same kind of graph. But in the case of iron, the

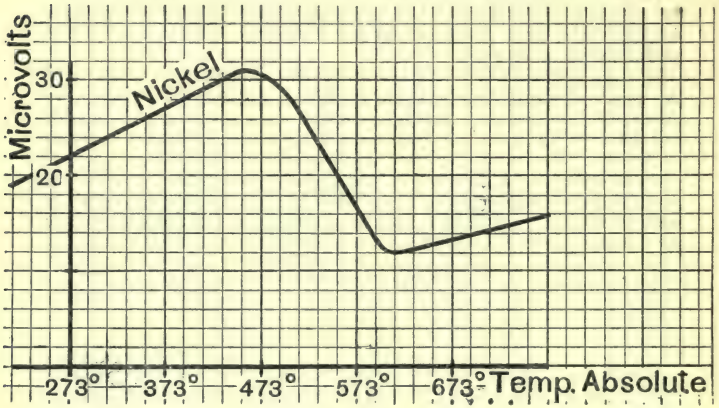


Fig. 126.

bends occur at higher temperatures than are shown in the figure. The first bend is near 800° C., where the magnetic properties show marked diminution. It is possible that many normal metals will prove abnormal at temperatures sufficiently extreme.

**249. Alloys.**—It is possible to produce alloys whose thermoelectric line have any desired slope. Imagine two metals (natural or artificial) to have lines exactly parallel;

one being at a uniform thermoelectric height $h$ above the other. Then the E.M.F. of a circuit of these metals, with junctions at temperatures $\theta$ and $\theta'$, will have E.M.F. $= h(\theta - \theta')$ proportional to the temperature difference. Such circuits are very exact and convenient measurers of temperature, within the limits for which the lines are straight. Alloys have often a low melting point; and when melting begins the physical properties alter. But a thermal couple of platinum and a platinum iridium alloy can be used up to the melting point of platinum. It is a question for experiment whether the usual formulae hold at such high temperatures.

**250. Compound Wires.**—Let the thermoelectric heights of two metals $P$ and $Q$ over a third metal $L$ be given by $a + bt$, $a' + b't$. Suppose wires of $P$ and $Q$ of resistances $p$ and $q$ to be joined in parallel, and this compound wire connected to a wire of $L$ of resistance $l$. Let $c$ and $c'$ be the currents in $P$ and $Q$; and $C = c + c'$ in $L$.

Then, by Kirchoff's laws,

$lC + pc =$ E.M.F. in the $L$ and $P$ circuit $= E$ say;

and

$lC + qc' =$ E.M.F. in the $L$ and $Q$ circuit $= E'$ say;

$\therefore \{l(p + q) + pq\} C = Eq + E'p.$

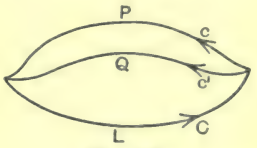

Fig. 127.

But $l + pq/(p + q) =$ resistance of the circuit formed by the single $L$ wire and the parallel $P$ and $Q$ wires.

Hence the apparent E.M.F. in this circuit

$$= \{l + pq/(p + q)\} C = (Eq + E'p)/(p + q).$$

Now if $t_1$ and $t_2$ be the temperatures of the junctions, and $t_0$, $t_0'$ the neutral points for $P$ and $L$, $Q$ and $L$, then

$$E = h(t_1 - t_2)(t_0 - \tfrac{1}{2}t_1 + t_2)$$

and

$$E' = h'(t_1 - t_2)(t_0' - \tfrac{1}{2}t_1 + t_2),$$

Hence $$\frac{Eq + E'p}{p + q} = \frac{bq + b'p}{p + q}\{t_1 - t_2\}\{T - \tfrac{1}{2}\overline{t_1 + t_2}\},$$

where $$T = (qt_0' + pt_0)/(p + q).$$

So that the gradient of the compound wire, relative to any metal $L$, is $(bq + b'q)/(p + q)$; where $p : q$ has any positive value. Hence given two metals we can combine them to get what is practically another metal of any chosen intermediate slope. It is naturally easier to get a compound wire whose line is parallel to the lead line or copper line than it is to prepare a reliable alloy of perfectly standard quality.

**251. Thermometry.**—We have seen (§ 249) that if two wires (natural or artificial) have parallel thermoelectric diagrams, the E.M.F. of a circuit is strictly proportional (within the limits of application of the formulae) to the difference of temperature.

If the two wires have their thermoelectric diagrams not parallel, we can measure temperatures with a little more difficulty. For, assuming the metals not abnormal, the E.M.F. is a quadratic function of either temperature (§ 245). Keep one junction at a fixed temperature, say 0° C.; and the observed E.M.F. will give a quadratic equation for the other temperature, of which one of the roots is often inadmissible. Thermal couples of highly refractory metals, as platinum, and iridium-platinum alloys, are used for temperatures of furnaces.

**Example.**—One junction is kept at 0° C. The E.M.F., in arbitrary units, is 30 and 45 when the other junction is at 100° and 200°. It is 35 when the junction is heated in a flame. Find the temperature.

Let $E$ be the E.M.F., $\theta$ the varying temperature of the junction. Since $E = 0$ when $\theta = 0$, we have $E = A\theta + B\theta^2$. To find $A$ and $B$ put $\theta = 100$, 200 and $E = 30$, 45; so that

$$30 = 100\,A + 10000\,B,$$
$$45 = 200\,A + 40000\,B,$$

whence $$A = \cdot375,\ B = -\cdot00075.$$

Solving for $\theta$ the equation

$$35 = \cdot375\theta - \cdot00075\theta^2$$

we have $\theta = 773°$. The other root is between freezing and boiling point, and obviously negligible.

Instead of solving this quadratic, we could have constructed a graph from the given data, with $E$ and $\theta$ as ordinate and abscissa. Such a graph would rapidly give the temperature for every E.M.F.

# CHAPTER XIV.

## THEORY OF UNITS.

**252. Units.**—Every physical quantity has its value expressed by a numeric and a unit. A *Numeric* is a number,* positive or negative, integral or fractional, rational or irrational. A *Unit* is a selected standard quantity of the same kind as the physical quantity whose value is expressed. Thus, if a charge be given as 5 coulombs, 5 is a numeric and coulomb is the name of a unit of charge. The operations of multiplication, involution, etc., can only be performed with numerics. When we write the dynamical equation

$$\text{Space} = \text{Velocity} \times \text{Time},$$

this is only a contracted way of expressing that the number of space units in the space traversed = the product of the number of velocity units in the velocity, and the number of time units in the time.

**253. Systems of Units.**—There is no necessary connection between units of different kind. It is even possible to have different units of the same kind. Thus a gallon and a cubic foot are both units of volume; a coulomb and an electrostatic unit of charge are both units of quantity of electricity.

But in order to simplify formulae superfluous units are abolished where possible, and the units retained are defined in terms of one another by the simplest possible

* Angles and Absolute Temperatures are also numerics. They are definable as ratios, and are the same in all systems of units.

relations. A set of units which satisfies these conditions is called a System.

In all systems, condensing the definitions, we have

Unit area = area of a square of unit side

„ volume = volume of a cube of unit side

„ density = such a density that there is unit mass in unit volume

„ velocity = such a velocity that unit space is described in unit time

„ acceleration = such an acceleration that unit velocity is gained in unit time

and so on.

**254. Fundamental Units.**—In dynamics, the units chosen as fundamental are those of length, mass and time. In physics, other fundamental units are adopted if necessary, such as units of electric charge, free magnetism, dielectric power or permeability. New units that have seemed fundamental drop out of use when they can be expressed in terms of other units; thus the calorie is not regarded as a fundamental heat unit, because it can be expressed as a simple multiple of the erg, the systematic unit of energy.

**255. Change of Units.**—Let us alter the fundamental units, in such a way that the new units of length, mass and time are $L$, $M$ and $T$ times the old units.

The new unit velocity has to be new unit length per new unit time (§ 253), so in terms of the old units it is that velocity with which a distance $L$ is described in time $T$. It therefore is $LT^{-1}$ old units of velocity.

The new unit acceleration produces a velocity $LT^{-1}$ old units in a time $T$; hence it is $LT^{-2}$ old units.

The new unit area is the area of a square of side $L$, it therefore is $L^2$ old units of area. Similarly the new unit volume is $L^3$ old units.

The new unit density is such that $M$ is the mass (in old units) of a volume $L^3$; so it is a density $M/L^3$ in terms of the old unit.

In the same way every new dynamical unit can be expressed as $L^a\ M^b\ T^c$ old units of the same kind.

**256. Dimensions.**—In the above formula, the indices $a$, $b$, $c$ are called the Dimensions of the unit considered in terms of the fundamental units of length, mass and time. The definition is as follows:—

If one only of the fundamental units be altered, by multiplying it by $x$; and the definition of the derived unit involves that it shall be multiplied by $x^r$, then $r$ is the dimension of the derived unit in terms of the fundamental unit considered.

If the new unit be $L^a\ M^b\ T^c$ old units, and we multiply only the length unit by $x$ leaving the mass and time units unaltered, then $L = x$, $M = 1$, $T = 1$; hence the new unit is multiplied by $x^a$, therefore $a$ is its dimension in terms of length, similarly for $b$ and $c$. The same definition applies to any other fundamental units that have to be adopted.

**257. Dimensions of Force.**—Let us define the unit of force as that which gives unit acceleration to unit mass. If the fundamental units be multiplied by $L$, $M$ and $T$, then the unit of acceleration is multiplied by $LT^{-2}$. The new unit force is that force which (in old units) gives acceleration $LT^{-2}$ to mass $M$. It therefore $= LMT^{-2}$ old units. Hence the dimensions of force are 1, 1 and $-2$.

If we had defined unit force as the weight, in Paris, of unit mass, the new unit force would be the weight of $M$ old units of mass, and therefore would be $M$ old units of force. The dimensions of force would therefore be 0, 1, 0 in length, mass and time. This system is preferred on the continent. It illustrates the fact that the dimensions of a quantity do not depend on the nature of the quantity, but on the way its unit is defined.

**258. Unit of Electric Charge.**—In any medium, the force $f$ between two point-changes $q$ and $q'$ is given by

$$f = \frac{1}{k} \cdot \frac{q\,q'}{r^2},$$

where $r$ is the distance apart, and $k$ the dielectric constant.

We shall define the units of charge and of dielectric constant in terms of one another, by the condition that $f = qq'/kr^2$ in every consistent system of units.*

This is clearly satisfied by the system of values $f, q, q', k, r$ each $= 1$. Let the units of length, mass, time, charge, and dielectric constant be $L, M, T, Q, K$ in terms of the corresponding units of another system. Then the equation must be satisfied when these other units are used, and therefore when $f = \mathrm{ML}/T^2$, $q = q' = Q$, $k = K$, $r = L$. Hence

$$ML/T^2 = Q^2/KL^2,$$
$$Q = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}.$$

**259. Unit of Field, Induction and Flux.**—*Field* is defined as force per unit charge. Hence the dimensions of field are symbolically represented by dividing those of force by those of charge; thus, using square brackets to denote a unit,

$$[\text{Field}] = [MLT^{-2}/M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}]$$
$$= [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}].$$

Induction is the product of field by dielectric constant, hence

$$[\text{Induction}] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{\frac{1}{2}}].$$

The surface integral of the Induction is got by adding together a number of products each of which is an induction multiplied by an area. Hence its dimensions are

$$[\text{Flux of Induction}] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{\frac{1}{2}} \times L^2]$$
$$= [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}].$$

Its dimensions are therefore the same as the dimensions of charge. This agrees with the fact that the surface integral of the induction over any closed surface is equal to the charge within it.

* This agrees with § 7 if $k = 1$ for air.

**260. Other Units.**—Other units may be considered more briefly. Current = charge per unit time,

$$\therefore \text{[Current]} = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}/T] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}}].$$

$$\text{[Potential]} = \text{[Energy per unit Current]}$$

$$= [ML^2 T^{-2}/M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$$

$$\text{[Capacity]} = \text{[Charge/Potential]} = [L K]$$

$$\text{[Resistance]} = \text{[Potential/Current]} = [L^{-1} T K^{-1}], \text{ etc.}$$

**261. Unit N. Pole.**—Let $m$ be the strength of a pole, then $4\pi m$ lines of force originate from it. Pole strength is not therefore analogous to point charge, since $4\pi q$ lines of induction, or $4\pi q\, k$ lines of force, originate from a charge $q$.

The tension along lines of force, or the pressure perpendicular to them, is $HB/8\pi = \mu H^2/8\pi$. So when the pole strengths of a system are given, and therefore the lines of force are determined, all the forces exerted are proportional to the value of $\mu$ for the medium.

Therefore the force exerted by $m$ on $m'$ at a distance $r$ is $\mu mm'/r^2$.

Hence **unit pole** is that which exerts on an equal similar pole at unit distance a force $1/\mu$. For the present, we leave $\mu$ undefined.

Using $P$ for the unit of permeability we deduce as in § 253 (replacing $K$ by $P^{-1}$)

$$\text{[Magnetic Pole]} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} P^{-\frac{1}{2}}.$$

**262. Other Units.**—From a pole $m$, $4\pi m$ lines of force diverge, hence the number of lines per unit area of a concentric sphere of radius $r$ is $4\pi m/4\pi r^2 = m/r^2$.

Hence Field = Pole strength/(Distance)$^2$

$$\therefore \text{[Field]} = M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} P^{-\frac{1}{2}}.$$

Similarly Induction = Field $\times \mu$

$$\therefore \text{[Induction]} = M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} P^{\frac{1}{2}}.$$

$$\text{Flux} = \text{Induction} \times \text{area} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} P^{\frac{1}{2}}.$$

Evidently also

$$[\text{Magnetic Moment}] = [\text{Pole} \times \text{Length}] = M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} P^{-\frac{1}{2}}.$$

$$[\text{Strength of Magnetic Shell}] = [\text{Magnetic Moment} \div \text{Area}]$$
$$= M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} P^{-\frac{1}{2}}.$$

**263. Current, and Related Units.**—The current in a circuit is equal to the strength of the equivalent shell, hence its dimensions are the same, therefore

$$[\text{Current}] = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} P^{-\frac{1}{2}}.$$
$$[\text{Charge}] = [\text{Current}] \times T = M^{\frac{1}{2}} L^{\frac{1}{2}} P^{-\frac{1}{2}}.$$
$$[\text{Potential}] = [\text{Energy} \div \text{Charge}]$$
$$= [ML^2 T^{-2} / M^{\frac{1}{2}} L^{\frac{1}{2}} P^{-\frac{1}{2}}]$$
$$= [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} P^{\frac{1}{2}}].$$

or

$$[\text{Potential}] = [\text{E.M.F.}] = [\text{Flux}/T]$$
$$= [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} P^{\frac{1}{2}}] \text{ as before.}$$
$$[\text{Capacity}] = [\text{Charge/Potential}] = [T^2 L^{-1} P^{-1}].$$
$$[\text{Resistance}] = [\text{E.M.F./Current}] = [L T^{-1} P].$$
$$[\text{Inductance}] = [\text{E.M.F./Rate of Change of Current}].$$
$$= [\text{E.M.F.} \times \text{Time/Current}] = [LP] \text{ etc.}$$

**264. Comparison of the Systems.**—Using different definitions of unit charge, we have reached different expressions for its dimensions in §§ 258, 263.

For a really scientific system the units are identical. Hence, equating them,

$$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}} = M^{\frac{1}{2}} L^{\frac{1}{2}} P^{-\frac{1}{2}}$$
$$\therefore PK = T^2/L^2.$$

So that when we change from one system to another, we cannot alter the units of permeability and dielectric power independently, for they are connected by an equation.

**265. Electrostatic and Electromagnetic Systems.**—Let $k$ be the dielectric power of air, and $\mu$ the permeability of air, in our standard system of units.

If we pass to another system by multiplying these units by $K$ and $P$, the dielectric power of air becomes $k/K$, and the permeability of air becomes $\mu/P$.

Suppose the units of mass, length, and time to be the centimetre, gram, and second. If we alter to the electrostatic system (defined by the fact that the permeability of air is 1) $k/K = 1$, $K = k$; and the units of length, mass, time are unaltered.

Hence the electrostatic unit charge

$$= M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}} = \sqrt{k} \text{ old units.}$$

In the same way if we change to electromagnetic units, defined by the convention that the permeability of air is 1,

$$M = 1, \quad L = 1, \quad T = 1, \quad P = \mu;$$

hence the electromagnetic unit charge

$$= M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} P^{-\frac{1}{2}} = 1/\sqrt{\mu} \text{ old units.}$$

Hence

$$\frac{\text{Electrostatic Unit Charge}}{\text{Electromagnetic Unit Charge}} = \sqrt{k\mu}.$$

**266. Velocity of Light.**—By many experiments it has been shown that the electromagnetic unit charge (absolute) $= U \times$ electrostatic unit charge, where $U = 3 \times 10^{10}$

$$= \text{velocity of light,}$$

either exactly or to a very close approximation.

Hence

$$U = 1/\sqrt{k\mu}, \quad k\mu = 1/U^2.$$

That is, the units of charge only agree if we adopt such units of $k$ and $\mu$ that the product of permeability and dielectric power of the standard medium air (or rather vacuum)

$$= 1 \div \text{square of the velocity of light.}$$

**Ex. 1.**—If $U$ be taken to represent the number of times an electromagnetic unit charge contains an electrostatic one, show that the ratio Electromagnetic unit / Electrostatic unit is $1/U$ for potential or E.M.F., $U^2$ for capacity, $U$ for current, $1/U^2$ for resistance and for inductance.

Using E.S. units throughout, the E.M. unit charge is the charge $U$. The E.M. unit potential is the potential to which the E.M. unit charge, *i.e.* the charge $U$, must be raised to do unit work ; it therefore is a potential $1/U$. The E.M. unit capacity is the capacity of a condenser in which a charge $U$ has potential $1/U$ ; it therefore is $U^2$. The E.M. unit current is a flow of $U$ units of charge per second, it therefore is a current $U$. The E.M. unit resistance is a resistance in which a potential difference $1/U$ gives a current $U$ ; it is $1/U^2$. The E.M. unit inductance is the mutual inductance of two coils such that a change of current per second of $U$ units in primary produces an E.M.F. of $1/U$ in secondary ; it therefore is $1/U^2$.

**Ex. 2.**—Show that [taking $K = 1$ and $P = 1$] the E.S. unit of resistance is a slowness and the E.M. unit is a velocity.

The dimensions are $[L^{-1} TK^{-1}]$ and $[LT^{-1} P]$. Neglecting $K$ and $P$ these are $[1 \div LT^{-1}]$ and $[LT^{-1}]$ as required.

**Ex. 3.**—Show that the E.S. unity of capacity is a length and the E.M. unit is the reciprocal of an acceleration.

**Ex. 4.**—A long stream of bullets is sent from a machine gun, with velocity $v$ at the rate of $n$ bullets per sec. If $e$ be the charge of each in electrostatic units, find the equivalent current in coulombs.

The current is $nev$ electrostatic units. It therefore is $nev/U$ electromagnetic units $= 10\,nev/U$ coulombs. Of course $U = 3 \times 10^{10}$ as above.

# CHAPTER XV.

## CORPUSCULAR THEORY.

**267. Units Employed.**—In this chapter we shall always imply the electrostatic system of units for charges, etc. Consequently the electromagnetic unit of charge will be called a charge of magnitude $U$ ($U = 3 \times 10^{10}$, see § 265), the E.M. unit of potential is a potential $1/U$, a coulomb is $U/10$, a volt is $10^8/U = \frac{1}{300}$. If $E$ be the charge carried in electrolysis by 1 gm. of hydrogen, its value is given in § 80 as 96500 coulombs. We shall therefore have

$$E = 96500\ U/10$$
$$= 2{\cdot}90 \times 10^{14}.$$

There is no need for greater accuracy, since the corpuscular data have not, as yet, been found certainly within 1 per cent.

**268. The Discharge in Rarefied Gases.**—At moderately low pressures, of the order of a centimetre of mercury, the discharge (if any) is a column of light. At lower pressures, below a millimetre, the column is seen to consist of alternate bright shells and dark spaces. Beginning from the kathode, each shell is of lower potential than the next one. Each can be regarded as a kathode in relation to the shell that follows it, or as an anode to the one that precedes it.

The first dark space, extending from the kathode to the first shell, is called the Crookes Space. The next, extending from the first bright shell to the next, is Faraday's.

As the rarefaction is increased, the Crookes space

extends until, with sufficient exhaustion, it reaches to the walls of the bulb. Such a bulb is called a Crookes Bulb. The glass bounding the Crookes space is rendered incandescent.

Fig. 128.

A small object (a mica cross) within the bulb casts a "shadow" on the glass, within which there is much less phosphorescence, precisely as though rays proceed from the kathode in straight lines. These rays must be partly (or wholly) stopped by the object. If the kathode be a considerably extended surface, the rays proceed from it normally. If therefore the kathode be concave, the rays converge to its centre of curvature. A solid placed at this point shows the most intense phosphorescence.

**Kathode Rays.**—The properties of the rays may be exhibited by a bulb as in Fig. 129. $K$ is the kathode;

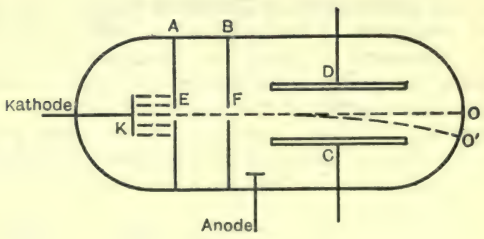


Fig. 129.

$A$ and $B$ are two screens with narrow slits at $E$ and $F$. These determine a narrow pencil of kathode rays $EOF$.

$C$ and $D$ are parallel metal plates whose purpose is described later.

The ray $EFO$ is impinging along its whole length on molecules of gas in the tube and therefore produces a certain amount of phosphorescence enabling it to be photographed from the side. Thus its straightness, under normal conditions, can be verified.

We shall show that the properties of the kathode rays can be explained by supposing them to consist of small particles negatively charged and travelling with a very high velocity.

**269. The $\beta$ Corpuscles.**—The particles which form the kathode rays will be called $\beta$ corpuscles, for reasons which appear later. We shall suppose them to have mass $m$, and velocity $v$, and to carry a charge (negative) of $-q$ electrostatic units.

If there be $n$ such corpuscles per unit length of a pencil of rays (produced as in § 268), the charge per unit length is $-nq$, the charge which passes in one second any section of the pencil (say the screen $B$) is $-nqv$. The pencil carries, therefore, a current $nqv$ (electrostatic units) towards the kathode.

**270. Effect of Electric Field.**—Let $C$ and $D$ in Fig. 129 be two parallel plates and let them be charged to a given potential difference so that there may be an electrostatic field $X$ acting upwards in figure, from $C$ to $D$. As each corpuscle is negatively charged, the force on it will be $Xq$ downwards.

As the force $Xq$ is perpendicular to the velocity, the path will no longer be a straight line, but will be curved with a radius of curvature $r_1$, such that

$$mv^2/r_1 = Xq. \qquad (1)$$

We need here only consider the case where the path is nearly straight, and $r_1$ is therefore large and uniform. If this limitation be abandoned the orbit would be a parabola, for we should have a particle moving (like a projectile) under a force constant in magnitude and direction.

**271. Effect of Magnetic Field.**—Let the magnetic field act perpendicular to the rays, and suppose it perpendicular to the plane of Fig. 129 and away from the reader. Let its magnitude be $H$ in electrostatic units, and therefore $H/U$ electromagnetic units.

The pencil is equivalent to a current $nqv$ E.S. units (§ 269), or $nqv/U$ E.M. units. Its direction is $OE$. Hence the force acting on unit length is in direction $DC$ and of magnitude $Hnqv/U^2$. The force per corpuscle is therefore $Hqv/U^2$. It would bend a ray, originally straight, into a curve whose radius of curvature is $r_2$, given by

$$mv^2/r_2 = Hqv/U^2. \tag{2}$$

If $H$ is large and constant, the orbit is a curve whose curvature is everywhere the same. If plane, it is an arc of a circle. If not plane, it is a helix, *i.e.* the curve of a corkscrew.

**272. Values of m/q and v.**—Dividing equation (2) by (1) we have

$$r_1/r_2 = Hv/XU^2, \tag{3}$$

whence $v$ is found if $r_1$ and $r_2$ be observed and $H$ and $X$ be known. The values of $v$ are different in different tubes and different intensity of discharge, but $v$ is of the order $U/10$, ranging up to $3U/10$ or $9 \times 10^9$ cm. per sec.

The value of $q/m$ can be got from (1) and (2) by eliminating $v$.

One of the methods employed by J. J. Thomson was to measure the deflection $OO' = x$ produced by the magnetic field, and then to adjust the electric field until the spot of phosphorescence produced on the glass was brought back to $O$. Then the forces due to $H$ and $X$ are equal and contrary, hence $r_1 = r_2$ and $= r$ say. Equation (3) becomes

$$Hv = XU^2, \; v = XU^2/H.$$

If the length $FO$ be $s$, evidently

$$OO' = x = s^2/2r, \tag{4}$$

hence $r$ is found; and equation 1, substituting for $U$, gives

$$q/m = v^2/rX = XU^4/rH^2. \qquad (5)$$

An approximate value of $q/m$ is $5{\cdot}3 \times 10^{17}$.

**Ex. 1.**—If $q/m = 6 \times 10^{17}$, and $v$ = one-tenth the velocity of light, find the electrostatic field, and the electromagnetic field, necessary to deflect a pencil 10 cm. long through a millimetre.

Here

$$S = 10,\ x = 1/10;$$

hence

$$r = s^2/2x = 500.$$

By (1),

$$X = \frac{m}{q} \cdot \frac{v^2}{r} = \frac{1}{6 \times 10^{17}} \cdot \frac{(3 \times 10^9)^2}{500} = {\cdot}03.$$

By (2),

$$H = \frac{m}{q} \frac{U^2 v}{r} = \frac{1}{6 \times 10^{17}} \cdot \frac{27 \times 10^{29}}{500} = 9 \times 10^9.$$

The electric field is therefore ·03, which can be reduced to volts per cm. by multiplying by 300, and therefore it could be produced between two plates a cm. apart by a potential difference of 9 volts. The magnetic field is $9 \times 10^9$ electrostatic units. Reducing to the more familiar electromagnetic units by dividing by $3 \times 10^{10}$, we get $H = {\cdot}3$. This is about two thirds the earth's resultant magnetic field.

**Ex. 2.**—If $\frac{q}{m} = 6 \times 10^{17}$, and $v = 3 \times 10^9$, calculate how much kinetic energy per sec. is carried by a current of $10^{-5}$ amperes.

Let $C$ be the current, in absolute E.S. units; and let $n$ = the number of corpuscles which pass per sec. Then $C = nq$.

The kinetic energy is $E = \frac{1}{2}nmv^2$.

Hence

$$\frac{E}{C} = \frac{1}{2}\frac{m}{q}v^2 = \frac{1}{2} \cdot \frac{9 \times 10^{18}}{6 \times 10^{17}} = 7{\cdot}5.$$

Now $10^{-5}$ amp. $= 10^{-6}$ E.M. units of current $= 10^{-6} \times 3 \times 10^{10}$ $= 3 \times 10^4$ E.S. units. Hence $C = 3 \times 10^4$ and $E = 2{\cdot}25 \times 10^5$ ergs.

**Ex. 3.**—If $q/m = 6 \times 10^{17}$ and $v = 3 \times 10^9$, show that the difference of potential between the kathode and the greater part of the bulb is about 2250 volts. Let the potential difference be $V$ absolute

units (electrostatic). The work done on each corpuscle is $Vq$, and this should = the kinetic energy gained = $\frac{1}{2}mv^2$. Hence

$$V = \frac{1}{2}\frac{m}{q}v^2 = \frac{1}{2}\frac{9 \times 10^{18}}{6 \times 10^{17}} = 7{\cdot}5.$$

To reduce to volts we multiply by 300, and we get 2250.

**Ex. 4.**—If $q/m = 5{\cdot}3 \times 10^{17}$ and $v = 2{\cdot}7 \times 10^9$ find the magnetic field which will make the corpuscle describe a circular path of radius 5 cm.

**Ex. 5.**—With the data of Ex. 4 find the electric field which would produce a curved path of radius 5 at the place where the field is normal to the path.

**Ex. 6.**—Corpuscles in a Crooks tube are incident perpendicularly at $O$ on a plane $XOY$, in which $OX$ and $OY$ are perpendicular coordinate axes. The corpuscles are then deflected simultaneously by a strong electric and a magnetic field, both parallel to the same direction $XO$; show that the electric deflection is parallel to $OX$ and the magnetic is parallel to $OY$. If they be called $x$ and $y$, find the locus of the point $(x, y)$ reached by the corpuscles for different values of $v$, assuming $q/m$ constant.

**273. Identical Nature of all $\beta$ Corpuscles.**—The preceding paragraph, and the examples in it, suggest methods by which $q/m$ can be found with more or less exactness. Professor J. J. Thomson and others have made most careful experiments on these lines, but the practical difficulties are very great and at first were nearly insuperable. It was comparatively easy to study the effect of magnetic fields (§ 271), but electric fields (§ 270) seemed to produce no effect. This was because the rarefied gas in the bulb was a fairly good conductor.

It was early established that the $\beta$ corpuscles are all of the same kind, independent of the nature of the gas, the material of the electrodes, and the intensity of the current. Hence $q$ and $m$ are constants [but see §§ 289-291].

**274. Atomic and Corpuscular Charge.**—It was regarded as practically certain that $q$ would turn out to be identical in magnitude to the charges $\pm e$ carried by an atom of H or Cl, or by any monovalent ion, in electrolysis.

Now 1 gm. of hydrogen carries a charge of 96500 coulombs, which is 9650 absolute E.M. units, or $9650 \times 3 \times 10^{10}$ E.S. units. If $M$ = mass of a hydrogen atom, it follows that

$$e/M = 9650 \times 3 \times 10^{10} = 2{\cdot}90 \times 10^{14}.$$

For the corpuscle, taking the value $q/m = 5{\cdot}3 \times 10^{17}$, and putting $q = e$,

$$e/m = 5{\cdot}3 \times 10^{17}.$$

Whence by division

$$M/m = 5300/2{\cdot}9 = 1800.$$

Hence the mass of a corpuscle is about 1 1800 of that of the lightest atom known.

The first determination of $e/m$ (J. J. Thomson in 1897) was $2{\cdot}3 \times 10^{17}$; but nearly every later experiment has given a higher result. The value in Kaye and Laby, *Physical and Chemical Constants* (1911), is equivalent to $5{\cdot}316 \times 10^{17}$.

**Ex. 1.**—Meyer (*Kinetic Theory of Gases*) estimates that the number of molecules per cubic cm. of a gas at normal temperature and pressure is $6{\cdot}1 \times 10^{19}$. Calculate from this the mass of a hydrogen atom, and of a corpuscle.

A c.cm. of hydrogen at normal conditions (0° C. and 1 atmosphere pressure) weighs $9{\cdot}0 \times 10^{-5}$ gm. Hence a molecule weighs

$$9{\cdot}0 \times 10^{-5} / 6{\cdot}1 \times 10^{19} = 1{\cdot}48 \times 10^{-24} \text{ gm.}$$

An atom is half this, and the mass of a corpuscle is

$$1{\cdot}48 \times 10^{-24}/2 \times 1800 = 4{\cdot}1 \times 10^{-28} \text{ gm.}$$

**Ex. 2.**—With same data, find the electronic charge.

The total charge for 1 gm. of hydrogen is $2{\cdot}9 \times 10^{14}$ E.S. units (§ 274). A cubic cm. weighs $9 \times 10^{-5}$ gm., hence its charge is $2{\cdot}61 \times 10^{10}$. The cubic cm. contains $2 \times 6{\cdot}1 \times 10^{19}$ atoms, hence the charge on each

$$= \frac{2{\cdot}61 \times 10^{10}}{2 \times 6{\cdot}1 \times 10^{19}} = 2{\cdot}14 \times 10^{-10}.$$

**Ex. 3.**—Taking a more recent estimate of the electronic charge, that it is $4{\cdot}7 \times 10^{-10}$, calculate the number of molecules per c.cm. of normal gas.

The answer is $2{\cdot}61 \times 10^{10} / 2 \times 4{\cdot}7 \cdot 10^{-10} = 2{\cdot}8 \cdot 10^{19}$.

**Ex. 4.**—If the P.D. between the anode and kathode is $V$ volts, show that $e$, $m$ and $v$ are connected by the approximate relation

$$eV = 150mv^2.$$

Find $v$ if $V$ is 2500 volts.

Note that nearly the whole change of potential takes place near the kathode. The P.D. in absolute units is $V/300$, hence the work done on each corpuscle is $eV/300$. This = the kinetic energy gained = $\frac{1}{2}mv^2$, whence the result.

Since $$e/m = 5{\cdot}3 \times 10^7,\ V = 2500,$$

$$\therefore\ 150v^2 = 1{\cdot}3 \times 10^{21},\ v = 3 \times 10^9.$$

**Ex. 5.**—By using a magnet to divert the kathode stream to fall on a small thermopile it is found that $h$ absolute units of heat (ergs) are conveyed per second. The current is $c$ electrostatic units. Show this gives a relation between $v$ and $e/m$.

If $N$ corpuscles impinge per second, their kinetic energy is $h = \frac{1}{2}Nmv^2$. The current is $c = Ne$. Hence dividing,

$$h/c = mv^2/2e.$$

The result, combined with equation (1), gives a value of $v$ and $e/c$. The result of Ex. 4 could be used the same way.

**275. Penetration of Matter by $\beta$ Corpuscles.**—We shall give reasons later for supposing that a $\beta$ corpuscle is enormously smaller than an atom. If it strikes an atom it usually penetrates it without much loss of kinetic energy. The corpuscles can therefore penetrate matter, even in its solid form, by passing between and through the atoms of which it is composed. Of course the penetration gradually destroys their energy. Different forms of matter oppose a resistance to the motion which is more or less, but not exactly, proportional to their density.

A moderate thickness of lead is almost impermeable; but aluminium is very permeable indeed.

The energy destroyed in a collision is transformed partly into visible forms of light, *i.e.* the phosphorescence observed on the walls of the bulb and along the path of the kathode rays (§ 268), largely into invisible light, and partly into Rontgen rays (§ 283).

**276. Lenard Rays.**—Lenard detected rays outside the Crookes bulb which were prolongations of the Crookes rays inside. These represent the relatively few corpuscles which pass through unstopped. To study them one should use a glass bulb with an aluminium end; for they get through aluminium fairly easily. These rays do not extend many centimetres in the atmospheric air outside the tube; for the velocity of the corpuscles is gradually diminished by collision with air molecules.

**277. Drifting Electrons.**—When a $\beta$ corpuscle has lost its excessive velocity through collisions with molecules, or through electric fields opposing its motion, it behaves almost like an ordinary gaseous molecule. The differences are only that it is very much smaller and that it carries a negative charge. It diffuses, like any other gas, but can be collected in any region of space by a feeble electric field.

Free $\beta$ corpuscles of small velocity can be produced in any gas, outside a vacuum bulb, by various stimuli (such as the incidence of ultra violet light on a metallic surface). Other free bodies positively charged, which we shall call $\alpha$ corpuscles, must also be produced; for it is obvious that whenever a negative corpuscle separates from an uncharged molecule, it must leave an equal positive charge somewhere or other. The nature of the $\alpha$ corpuscles is considered later.

These corpuscles may be said to drift, because they have not the enormous directed velocities met with in Crookes tubes. But they are by no means inert; even an uncharged hydrogen molecule has an average velocity of about $2 \times 10^5$ cm. per sec. if at normal temperature.

Apparently, however, a charged electron can attract and attach to itself a number of uncharged molecules. It then becomes a charged complex, which is heavier than an ordinary molecule and diffuses slower, but still has a reasonably high velocity and can be sorted out rapidly enough.

Gases loaded with electrons can have them entirely filtered out by passing through cotton-wool. This is not, of course, because the electrons are too large to go through

a filter, but because their charge makes them drift to the sides of any narrow channel, and adhere.

**278. Conduction of Electricity.—In electrolytes** we have seen that conduction is due to the transport of $+^{ve}$ and $-^{ve}$ charges by atoms or molecular aggregates called irons. These drift under the influence of electric fields. In an electrolyte (*e.g.* dilute HCl) which is not exposed to an electric field, the ions ($H+$ and $Cl-$) move aimlessly like ordinary molecules and exert an osmotic pressure of exactly the same kind.

**In gases** in their normal state there are hardly any charged ions, and consequently gases hardly conduct at all. But certain agencies (see § 277) ionise gases, and they then show conductivity. The conduction is due to the transport of + and − charges by carriers, which are $\alpha$ and $\beta$ corpuscles, free or loaded, as above, with uncharged molecules.

**In metals** it is believed that electricity is carried almost entirely by $\beta$ corpuscles (negative), which travel promiscuously in all directions if there be no electric field. An electric field slightly assists motion contrary to its direction, and therefore a small excess of $-^{ve}$ electricity flows contrary to the field. There is evidence, however, that positive travelling ions may exist.

**279. Loss of Conductivity of Gases.**—The positive and negative carriers or ions, drifting with considerable velocity and attracting one another, have a tendency to seek each other out and coalesce into neutral bodies, *i.e.* into ordinary gaseous molecules. Thus a gas, made conducting and then left to itself, loses its conductivity at a rate proportional to the number of ions present, and therefore proportional to the conductivity itself.

**280. Measurement of Conduction in Gases.**—The conductivity is so small in a normal gas, even when fully ionised, that none of the usual electromagnetic methods are of any use. The current will not affect a galvanometer. But one can find the leakage between a metal disc con-

nected with an electrometer, and a parallel plate earthed, or kept at a different potential. For method see Ex. 1 below.

Let $N$ be the number of ions produced per second, by any of the agencies described in § 277 ; half of them are $+^{ve}$ and half are $-^{ve}$. If the field be exceedingly strong, the ions can reach one plate or the other in a time too short for them to be appreciably diminished in number; so that $\frac{1}{2}N$ reach each plate per second, and a charge $+\frac{1}{2}Ne$ reaches the kathode while $-\frac{1}{2}Ne$ reaches the anode. Thus the current is $\frac{1}{2}Ne$, independent of the area of the plates or their distance apart. Thus the greatest current possible depends only on the ionising power of the agent.

**Ex. 1.**—A sensitive electroscope consists of a gold leaf repelled by a fixed metal plate to which it is connected. The position of the leaf is read by a microscope on a scale calibrated in volts. The disc of the electroscope leaks across an air gap to another disc which is earthed. If the electrostatic capacity of the electroscope be six and the fall of potential be one volt per minute, calculate the current carried by the air.

The potential falls one volt, *i.e.* 1/300 E.S. units, per minute, and the capacity is six ; hence the charge that escapes per minute is $6/300 = 1/50$, and the current is 1/3000 E.S. units per second.

An E.S. unit is $1/3 \times 10^{10}$ E.M. units ; or $1/3 \times 10^{9}$ coulombs. We can therefore describe the current as $1{\cdot}9 \times 10^{13}$ E.M. units, or $1/9 \times 10^{12} = 1{\cdot}1 \times 10^{-13}$ amperes.

**Ex. 2.**—If, in the above example, the air be continuously kept ionised to the same extent, and the electroscope were originally charged to 500 volts, how would the current alter with the time?

As the current carried does not depend on the potential difference (if it be considerable), and only depends on the carriers available, the current would remain sensibly constant for nearly 500 minutes, and the electroscope would then be nearly uncharged. However, a limit would be reached when the potential difference reached a certain degree of weakness. For the time taken by the carriers to travel across is inversely proportional to the field which urges them. If this time exceeds the lifetime of a carrier, *i.e.* the time which it lasts before it combines with one of the opposite kind, and becomes inert, then no current is carried.

**Ex. 3.**—What would happen if, in Ex. 1, the gas were ionised at start, but not kept ionised?

The carriers would be destroyed both by acting as conductors and by coalescing with the opposite kind. Hence conduction would cease after a time.

**281. The Number of Corpuscles in a Given Space.**—If perfectly dust-free air be left in a closed vessel in contact with pure water at any temperature, it becomes saturated with water vapour. If the air space be suddenly increased by lowering the water level, the vapour pressure is diminished, but the temperature also falls (because the change is adiabatic) to such an extent that the diminished vapour pressure is too great for saturation. There is no cloud formed, since there is nothing to deposit water on. We have clear super-saturated water vapour, which only very slowly deposits on the water surface and the sides of the vessel.

But if the dust-free air contain ions, a sufficient super-saturation causes deposition to take place on the negative corpuscles, and a more extreme super-saturation causes deposition on the positive particles likewise.

Let the super-saturation be controlled so that only the negative corpuscles are affected. The mass $M$ of condensed water vapour per cubic cm. can be calculated from the degree of super-saturation permitted, or can be directly found by weighing the cloud. Let $a$ = radius of each water drop, $m = \frac{4}{3}\pi a^3$ = volume or mass of the drop, and $N = M/m$ = number of corpuscles per c.cm.

By a formula due to Stokes, if $\beta$ be the coefficient of viscosity of air, the drops will fall with a sensibly uniform velocity $w = 2ga^2/9\beta$ [$g = 981$]. The rate of fall $w$ is not hard to measure, and thus $a$ is found.

Now the volume of each drop in c.cm., or its mass in grams, is $\frac{4}{3}\pi a^3$. But the whole mass of the water formed into drops can be determined, either by simply weighing the cloud or, if it be produced within a closed space, by calculating from the change of vapour pressure the amount of water which must have been precipitated.

**282. The Absolute Charge and Mass of a Corpuscle.**—Let the positive corpuscles be separated from the negative ones by means of an electric field. Let water have been deposited on the negative ones, as in the previous paragraph; and their velocity of fall $w$ be observed.

Apply to the descending cloud an electric field of magni-

tude $X$ (E.S. units), directed downwards. Each drop contains a negative corpuscle of charge $-e$.* Hence an upward force $Xe$ acts on each drop, and the cloud is suspended quite stationary if $Xe =$ the weight of the drop $= \frac{4}{3}\pi a^3 g$. The cloud can be made to rise or fall by increasing or diminishing $X$.

By observing $w$ we find $a$ and by observing $X$ we then find $e$. If $m$ be the mass of a corpuscle, $m$ is found from the equation $e/m = 5{\cdot}3 \times 10^{17}$.

Assuming that the corpuscular charge $e$ is also the charge of a monavalent atom in electrolysis, and knowing the charge carried by a gram of hydrogen, we get the mass of a hydrogen atom.

J. J. Thomson gave the earliest reliable measurements. A mean derived from the latest results (Kaye and Laby) is

$$e = 4{\cdot}7 \times 10^{-10} \text{ E.S. units.}$$

Hence the mass $m$ of the $\beta$ corpuscle is

$$4{\cdot}7 \times 10^{-10}/5{\cdot}3 \times 10^{17} = 8{\cdot}9 \times 10^{-28}.$$

A gram of hydrogen carries the charge $2{\cdot}90 \times 10^{14}$, while a hydrogen atom carries $e$. Hence the number of hydrogen atoms in a gram is

$$2{\cdot}90 \times 10^{14}/4{\cdot}7 \times 10^{-10} = 6{\cdot}16 \times 10^{23}.$$

The mass of a hydrogen atom is

$$1/6{\cdot}16 \times 10^{23} = 1{\cdot}62 \times 10^{-24}.$$

**Ex. 1.**—If the coefficient of viscosity of air be ·00017, find the radius and mass of a water drop which descends at the rate of ·04 cm. per second.

Here $w = {\cdot}04$, $\beta = {\cdot}00017$; but $w = 2ga^2/9\beta$. Hence $a^2 = {\cdot}04 \times 9 \times {\cdot}00017/2 \times 981$; giving $a = {\cdot}000177$. The mass $= \frac{4}{3}\pi a^3 = 2{\cdot}3 \times 10^{-11}$.

**Ex. 2.**—If the cloud was rendered stationary by an electric field of 74 units (E.S.), find $e$, the negative charge per corpuscle. Evidently

$$e \times 74 = 2{\cdot}3 \times 10^{-11} \times g,$$

hence

$$e = 2{\cdot}3 \times 10^{-11} \times 981/74 = 3{\cdot}05 \times 10^{-10}.$$

* It was found that a few drops contained two, or even three, corpuscles; and these behaved rather differently.

**Ex. 3.**—Estimate the number of molecules per cubic centimetre of any gas at normal pressure and temperature from data of § 282.

The number is the same as for hydrogen. The mass of a cubic cm. of hydrogen is $9 \times 10^{-5}$ gm. The number of atoms of hydrogen is got by dividing by the mass of an atom, and is

$$9 \times 10^{-5}/1{\cdot}62 \times 10^{-24} = 5{\cdot}6 \times 10^{19}.$$

The number of molecules is half this,

$$= 2{\cdot}8 \times 10^{19}.$$

**Ex. 4.**—With data as in Ex. 1, 2, find the mass of a globe of water which descends at the rate of 1 cm. per min. in air; and the electric field which will make it ascend at this rate.

**Ex. 5.**—Find the number of molecules of a gas in a mol.

**Ex. 6.**—Assuming that the radius of a gaseous molecule is $10^{-8}$, compare the volume of the molecule itself to the volume per molecule in a gas at normal temperature and pressure.

**Ex. 7.**—Find $p$, the "mean free path" of a gaseous molecule between encounters with other molecules; assuming that $4\pi a^2 p = v$ where $a$ is radius of molecule, and $v$ is the volume per molecule. Find also $p'$, the mean free path of a corpuscle between encounters with the molecules of a gas, assuming $\pi a^2 p' = v$. Take the gas at normal temperature and pressure.

**283. Rontgen Rays. X-rays. $\gamma$ Rays.**—These have already been alluded to. They are given off whenever a $\beta$ corpuscle, moving with a sufficiently high velocity, strikes matter.

They are not deflected by any electric or magnetic field, and are therefore not supposed to consist of travelling charged particles. The natural hypothesis is that they are disturbances in the ether, similar to light waves. But while light travels in a long series of similar waves, the Rontgen rays appear to be either very short trains of waves or, more likely, isolated disturbances.

They originate from every portion of a Crookes or Lenard stream, being caused by the corpuscle striking gaseous molecules; but they originate most abundantly from the point (as $O$ in Fig. 129) where the rays strike a solid. They are not regularly reflected or refracted, owing to the shortness of the wave-series. As is well known,

they affect photographic plates, and phosphorescent screens. They easily traverse a considerable thickness of matter, being much more penetrative than $B$ corpuscles. Hence they can be used in surgery to examine the bones and viscera of a living person.

**284. Canal Rays, or $\alpha$ Rays.**—These are harder to detect in vacuum tubes than Crookes radiation, though they occur as universally. They consist of positive electrons, which travel towards the kathode. If the kathode be a short metal hollow cylinder, many of these will rush down the cylindrical canal, and stream backwards behind the kathode. Their velocities and ratio $m/e$ can be observed in the same way as for the $\beta$ rays; but with more difficulty. The velocities are much smaller (about $10^8$ cm. per sec.); and $m/e$, in various cases, is either the same as for a hydrogen atom or twice or four times as great. In fact, people have spoken of three types of $\alpha$ electrons as hydrogen atoms, hydrogen molecules, and helium atoms.* But, likely as this may be, none can yet say that it is fully justified. The $\alpha$ corpuscles of radium are probably helium atoms with charge $2e$, so that $m/e$ is twice that for hydrogen.

**Ex. 1.**—If an $\alpha$ corpuscle be a charged hydrogen atom, what velocity is given to it by a P.D. of 2500 volts?

Precisely as in Ex. 4, § 274,

$$eV = 150\, mv^2.$$

But for hydrogen $e/m = 2{\cdot}9 \times 10^{14}$;

$$\therefore \quad 150v^2 = 2{\cdot}9 \times 2500 \times 10^{14}; \quad v = 7 \times 10^7.$$

**Ex. 2.**—Taking $e/m = 2{\cdot}9 \times 10^{14}$, $v = 10^8$, for $\alpha$ rays, and $e/m = 5{\cdot}3 \times 10^{17}$, $v = 3 \times 10^9$ for $\beta$ rays, calculate which is most affected by a magnetic field.

The radius of curvature of path produced by a field of $H$ (electromagnetic) units is given by

$$mv^2/r = Hev, \text{ or } U^2.$$

Hence $U^2/Hr = e/mr$, $= 2{\cdot}9 \times 10^6$ for $\alpha$ rays, and $1{\cdot}8 \times 10^8$ for $\beta$ rays. Consequently the curvature of path is 60 times as great in

* The atomic weight of Helium is 4. The molecule of helium is a single atom.

the latter case. This explains why it is so much harder to make experiments on the magnetic deflection of canal rays than on that of kathode rays.

[For the other rays in which $e/m$ is half or quarter as large, the curvature is still smaller.]

**Ex. 3.**—Are $\alpha$ or $\beta$ rays more affected by electric fields?

Here the curvature is proportional to $e/mv^2 = 2{\cdot}9 \times 10^{-2}$ for $\alpha$, and $5{\cdot}5 \times 10^{-2}$ for $\beta$. The $\beta$ rays are more affected but the difference is not very great.

**285. Function of the $\alpha$ Corpuscles.**—In a Crookes tube the $\beta$ corpuscles are much more obvious than the $\alpha$ ones, and carry practically the whole discharge in virtue of their greater velocity. But the $\beta$ corpuscles appear to owe their existence to the violent impact of the much more massive $\alpha$ corpuscles against the kathode. If part of the kathode be screened, no Crookes radiation originates from the screened portion.

**286. Radioactivity.**—Ordinary radium sends out rays of the three kinds called $\alpha$, $\beta$, $\gamma$ which have since been identified with Canal, Crookes, and Rontgen rays. If a little radium be put at the bottom of a narrow fissure in a block of lead (see Fig. 130), the radiation is hindered in all directions but one, and the emerged stream can be analysed by a powerful magnetic field (perpendicular to plane of figure, and towards observer). This stream deflects the positive current, or $\alpha$ rays, very slightly to left (see Ex. 2, § 284); the negative current, or $\beta$ rays, considerably more to the right, and leaves the $\gamma$ rays undeflected.

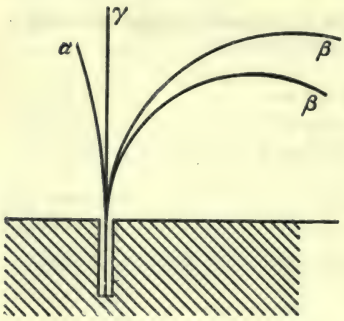

Fig. 130.

The names $\alpha$, $\beta$, $\gamma$ rays were first used in describing the properties of radium.*

**287. Mass and Inertia.**—Mass is generally defined as Quantity of Matter. It is therefore axiomatic that two identical objects have double the mass of one of them, and that the mass of an object is the same whether it be at rest or in motion with any velocity. It is not practicable to compare masses of different objects by dividing them into identical parts which can be counted. Hence masses are compared either by weighing, or by testing their inertia. The first process, utilising the fact that the earth attracts equal masses equally at the same place, cannot be applied to corpuscles. The second process depends on the fundamental principles of mechanics, which lead to the equation,

$$\text{Acceleration} = \text{Force}/\text{Mass}.$$

If therefore equal forces be applied to different masses, the effects of these forces (measured by the accelerations produced) are inversely as the masses, and do not depend on anything else. . In particular, the effects do not depend on the velocity already possessed by the mass acted upon.

**288. Apparent Failing Case of Dynamical Equations.**—If we have a body moving in a medium, *e.g.* a bullet travelling in air, any motion of the body moves the medium as well. Hence a force applied to a bullet in air will produce rather less acceleration than if it acted on the bullet in vacuo.

If the bullet is travelling at an enormous speed, it carries more air with it, and its effective mass is increased.

There are, of course, many differences between the motion of a bullet in the air and of a corpuscle through the ether. Air is a compressible gas, ether appears to be

* Its state of chemical composition or mechanical aggregation is immaterial. Radium bromide is generally used. The phase "ordinary radium" signifies the usual aggregate of radium proper, the first emanation, and Radium *A*, *B*, *C*, *D*. It is distinguished from true radium, from which the emanations have been removed (see § 295).

quite incompressible, and the vibrations which it transmits are transverse, while sound waves in gases and liquids are longitudinal. But there is this resemblance, that the apparent mass of a corpuscle increases at high speeds. Theory shows that the apparent mass ought to become infinite when the velocity $v =$ that of light.

**289. Electric Inertia.**—Instead of considering motions of the ether, about which we know nothing definite, consider the electrical phenomena to which they are equivalent.

A $\beta$ corpuscle, of charge $-e$ and mass $M$, moving with velocity $v$, is equivalent to a portion of an electric current $ev$. This has a certain self-inductance $L$, and therefore has kinetic energy $\frac{1}{2}L(ev)^2$; whilst its mechanical kinetic energy is $\frac{1}{2}Mv^2$. Hence the total kinetic energy is

$$\tfrac{1}{2}(M + m')v^2$$

where $m' = Le^2$, and can be called the apparent extra mass due to the charge, or simply the electric inertia. The total inertia is the sum of the mechanical and electric inertia. Call it $m$, then $m = M + m'$.

**290. Calculations of the Electrical Inertia.**—On the supposition that the corpuscle is a small sphere of radius $a$, its charge being in a superficial layer of uniform surface density, J. J. Thomson calculated the electrical inertia, at low speeds, to be

$$m' = \frac{2}{3}\frac{e^2}{aU^2}.$$

Thomson, and others later, have given formulae suitable for high speeds. These formulae agree qualitatively, $m'$ becoming infinite when $v = U$; and their difference is due to different assumptions made and the different meaning attached to $m'$.

Suppose a corpuscle to have mechanical mass or inertia $M$, and electrical inertia $m'$, so that $M$ is independent of the speed. The apparent inertia $m = M + m'$; we suppose $m$ defined by the dynamical equation

$$\text{Force} = \text{acceleration} \times m.$$

We have seen how to find $v$ and $m/e$ (§§ 272, 274); and $e$ can be considered constant. The $\beta$ corpuscles emitted by radium can have very high speeds, approaching $U$ or $3 \times 10^{10}$. The value of $m/e$ for any speed $v$ can be expressed as a multiple of the sensibly constant value of $m/e$ for such low speeds as are found in a Crookes tube (about $U/10$).

**291. Ratio of Mechanical to Electrical Mass.**—Kaufmann investigated the values of $e/m$ for radium $\beta$ corpuscles of different high velocities. Of course $e$ is an absolute constant, and the $m$ observed is constant at low speeds; call it $m_0$. The observed $m$ at higher speeds was expressed as a multiple of $m_0$, thus $m/m_0$ was tabulated as a function of $v$.

For comparison, he calculated the electrical mass $m'$ according to one of the formulae which expressed it as a function of $v$.

It was found that $m$ and $m'$ altered in the same ratio (within small experimental errors). It is inferred that the mechanical mass $M$ is either zero or at least very small. Assuming it zero, we should conclude that the mass of a corpuscle is altogether electrical.

Hence the phenomenon of mass is due entirely to the electrical phenomenon of self-induction. If this be true (as is probable), it represents a great scientific advance: for two apparently different properties are reduced to one.

**292. Radius of the $\beta$ Corpuscle.**—If the whole mass is electric, then $m'/e$ = the value given for $m/e$ in § 272 $= 1/5{\cdot}3 \times 10^{17}$, for slow speeds.

But assuming Thomson's formula (§ 290)

$$m' = \tfrac{2}{3} e^2 a U^{-2}.$$

Hence

$$\tfrac{2}{3} e/a U^{-2} = 1{\cdot}5{\cdot}3 \times 10^{17}.$$

Putting

$$U = 3 \times 10^{10},$$

and

$$e = 4{\cdot}7 \times 10^{-10},$$

we get

$$a = 1{\cdot}85 \times 10^{-13}.$$

The radius of a hydrogen atom is about $10^{-8}$; so evidently the corpuscle is very much smaller.

The density of the corpuscle, got by dividing the mass ($8{\cdot}9 \times 10^{-28}$, see § 282) by $\frac{4}{3}\pi a^3$,

$$= 3{\cdot}4 \times 10^{10}.$$

Of course this makes the unverified assumption that the mass belongs to the portion within the radius $a$. But, as Oliver Lodge has pointed out, this calculation gives an idea of the order of magnitude of the density of the luminiferous ether. He obtains a larger result by using a different value of $a$ (*The Ether of Space*, App. 2). Since ether pervades everything (and in fact *is* everything), and since it is incompressible, its density cannot be less than that of the densest thing known.

**293. Provisional Idea of the Nature of an Atom.**—Practically the whole inertia or mass of the atom is supposed to belong to its negative corpuscles. Of these, there are roughly 1800 in the case of a hydrogen atom (§ 274); and more for other atoms in the ratio of their molecular weights. The positive electricity of an uncharged atom is equal in quantity to the negative; it is not concentrated on corpuscles but diffused uniformly over a more or less spherical positive cloud, of practically no mass. Within this cloud, the attraction on a corpuscle is towards the centre and proportional to the radius; consequently the corpuscles have constant periods independent of amplitude. The corpuscles are assumed not to travel separately (like the different planets of the solar system) but to form a set of chains or series, the particles in each chain following one another in an orbit that can be regarded as circular. The distances between corpuscles are enormous compared to the corpuscles themselves; in fact the corpuscles within an atom have been likened to "bees in a cathedral." The estimates given (§ 292) of radii of atom and corpuscle are in ratio of the order $10^5 : 1$ (one and a half miles to an inch; a kilometre to a centimetre).

These ideas must not be regarded as facts. If they should ever be confirmed they are sure to be also very

much modified. It is exceedingly difficult to deduce from these theories any mathematical consequences which can be experimentally tested. Some of the experimental evidence that exists belongs to Optics [*e.g.* the Zeeman Effect of a magnetic field on the spectrum of an element]. Of the corpuscles, the great majority are stable; but a few (supposed not to exceed 8) are more loosely connected and may be temporarily lost. An atom liable to lose one $\beta$ corpuscle can obviously act as a monovalent positive element. An atom liable to annex a corpuscle is monovalent negative. Chemical relationships are thus governed by the number of disposable corpuscles. Chlorine can annex a corpuscle and is the negative monovalent element in HCl, and chlorides generally. But it can part with five or perhaps seven corpuscles and is the positive element, with valency 5 or perhaps 7 as a maximum, in the compounds of chlorine and oxygen.

**294. Unstable Atoms.**—We have seen that even a hydrogen atom is a very complex structure. The atoms of elements of great atomic weight [Uranium, Radium, Thorium] so nearly approach instability that they occasionally break into simpler forms.

An atom so altering may, or may not, detach a $\beta$ corpuscle, or an $\alpha$ corpuscle, travelling at a high speed. Slow moving corpuscles, if produced, would probably not be detected.

If rapid $\beta$ corpuscles are liberated, there must also be $\gamma$ rays; for these originate wherever swift $\beta$ corpuscles impinge on matter.

We have seen that "ordinary radium" gives both $\alpha$ and $\beta$ corpuscles, and therefore $\gamma$ rays.

**295. Transformations of Radium.**—An atom of radium can spontaneously give off an $\alpha$ corpuscle: it then alters into a substance called **Radium Emanation.** Let $T$ seconds be the average lifetime of a radium atom as such; this means that, of $N$ radium atoms, in every second $N/T$ will turn into emanation, and $N$ $T$ $\alpha$-corpuscles are emitted. The quantity of radium present is multiplied

by $1 - 1/T$ in each second; its logarithm decreases by $-\log(1 - 1/T) = 1/T$; and the quantity of radium is halved in a time $\log_e 2 \div (1/T) = \cdot 693T$. The time in which a quantity of radium halves itself has deduced from observation, and is about 1200 years [Rutherford], so $T = 1200/\cdot 693 = 1700$ years roughly.

The emanation is a gas. Its atom can spontaneously give off an $\alpha$ corpuscle. Its average lifetime $T_1$ is much less than $T$. Its quantity halves itself in 3·8 days, therefore $T_1 = 3\cdot 8/\cdot 693 = 5\cdot 5$ days. The emanation produced within the radium salt does not readily escape; consequently ordinary radium consists of radium with emanation and later derivatives. For equilibrium, the emanation is destroyed as fast as it is produced; therefore

$$\frac{\text{Quantity of emanation}}{T_1} = \frac{\text{Quantity of radium proper}}{T},$$

$$\therefore \frac{\text{Quantity of emanation}}{\text{Quantity of radium}} = \frac{T_1}{T} = \frac{3\cdot 8}{1200 \times 365} = \frac{1}{110000}.$$

The emanation is only occluded. It can be separated by dissolving and heating the radium salt.

The emanation atom, losing an $\alpha$-corpuscle, becomes Radium A. This halves its quantity in 3 minutes. Radium A, losing an $\alpha$-corpuscle, becomes Radium B with a time constant 21 minutes.

The changes are tabulated [from Rutherford, *Radioactivity*] as under:—

| Product. | Time to be half transformed. | Rays. |
|---|---|---|
| Radium | 1200 years | $\alpha$ |
| Emanation | 3·8 days | $\alpha$ |
| Radium A | 3 minutes | $\alpha$ |
| ,, B | 21 minutes | none |
| ,, C | 28 minutes | $\alpha$, $\beta$ (and $\gamma$) |
| ,, D | 40 years | none |
| ,, E | 6 days | $\beta$ (and $\gamma$) |
| ,, F | 143 days | $\alpha$ |
| ,, G | ? | |

In old radium, each constituent is produced as fast as destroyed and therefore every product is present in quantity proportional to its time constant. But if the emanation, etc., be removed and we start with radium proper, it will be a few days before the emanation and *A*, *B*, *C*, *D* are in steady quantity, and one would have to wait many years before the quantity of *E*, and therefore *F* and *G*, stopped sensibly increasing. It is suspected that Radium *G* is lead.

**296. Other Radioactive Substances.** — Uranium, Thorium, Actinium, Polonium show properties like Radium. The radioactive substance Polonium is possibly not a member of a distinct series, but the same as Radium F. The time constant and the nature of the rays are the same for both.

As Radium (so far) is only found in Uranium ores, and in a nearly fixed ratio 1 : 220, it is inferred [Rutherford] that Radium is a transformation product of Uranium, and that the average lifetime of Uranium is about 220 times that of Radium, or 400,000 years.

If Radium were not a product of some body with a much longer life, it would be hard to account for its presence on the earth in measurable quantities.

A mass of radium equal to that of the earth would reduce, in about 150,000 years, to a practically equal mass of Radium G, and a single gram of radium. This time is a mere trifle compared to the age of the earth, and *a fortiori* to that of the universe.

It is possible that all atoms are unstable. But it is quite certain, in most cases, that their average life is very long compared even with that of Uranium.

**Ex. 1.**—If the quantity of radium is halved in 1200 years, in how long time would a ton of radium become a gram?

A ton almost = a tonne, or a thousand kilograms = $10^6$ gm. If the time be $t$ years, the quantity is halved $t$ 1200 times.

Hence

$$(1/2)^{t/1200} = 1/10^6,$$

$$t = 1200 \log 10^6 / \log 2 = 24000.$$

**Ex. 2.**—If the earth's mass be $6 \times 10^{21}$ tons, how long would this take to reduce to a ton?

Evidently

$$t = 1200 \{\log 6 + 21\}/\log 2 = 87{,}000.$$

**Ex. 3.**—How long would it take for a ton of pure uranium to diminish to a gram? How much radium would be present in the ton of remaining matter?

**Ex. 4.**—If the atomic weight of radium is 226, and the $\alpha$ corpuscle is double the hydrogen atom, find the loss of weight per year of a gram of ordinary radium as a result of emission of $\alpha$ corpuscles. Note, of course, that the resultant of all the changes is that radium proper becomes radium $D$, and four positive corpuscles.*

**Ex. 5.**—If a gram of ordinary radium is enclosed in lead so thick as to prevent the escape of $\alpha$ corpuscles, but not of $\beta$, show that the radium accumulates a positive charge. Find the rate at which charge is accumulated per year.

**Ex. 6.**—It is found that ordinary radium (practically free from $D$, $E$, $F$, $G$) produces heat energy at the rate of 100 calories (nearly) per gram per hour. Assuming that each atom gives ultimately $4\alpha$ corpuscles, and that the average lifetime of a radium atom is 1700 years, find the average energy given to each corpuscle as a multiple of the corpuscular mass. If we suppose energy $= \frac{1}{2}mv^2$, show this implies an average velocity of about $2 \times 10^9$ cm. per second.

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* Later changes, up to $G$, are neglected because $D$ alters so slowly that they would not be apparent for several years.

# ANSWERS.

§ 3.—**Ex. 3.** $\sqrt{3}+1 : \sqrt{3}-1$, or 3·73 : 1. **4** 25/16, = 1·56.

§ 5.—**Ex. 1.** 145°. **2.** 24°, 130°.

§ 9.—**Ex. 1.** $4\sqrt{3}$ perpendicular to $OA$. **2.** ·566 at centre: ·0142, ·0436 at mid points.

§ 13.—**Ex. 3.** 0, 0, $\pm\{2/5 - 2/5\sqrt{5}\} = \pm$ ·221.

§ 34.—**Ex. 7.** Proportional to $bc/(br + cr + bc)$ and two similar quantities. **8.** Charges are $\pm$ 9/10.

§ 47.—**Ex. 1.** 1/16 dyne. **3.** 3/32 dynes, nearly $25\frac{1}{2}°$ to normal. **5.** $\pm$ ·0407, 0; ·0415, ·0291. **6.** $-1/5\pi = -$ ·06438, $-1\cdot1440\pi = -$ ·00044; ·0608 dyne. **7.** ·0621 dyne. **9.** Densities in 6 would be altered by $+1/140\pi$ or + ·00023; ·0322 dyne. In 7, the force is $35/24^2 + 35/74^2 - 10/343 - 1/196 =$ ·0331 dyne. **10.** $1\cdot10\pi =$ ·0318; $1/640\pi =$ ·0005. **11.** With figure of Ex. 3 the potentials at $A$ and $B$ are each $V = -1/4\sqrt{2}$, hence the mutual energy $E = -\frac{1}{2}\Sigma QV = -1/4\sqrt{2} =$ - ·177 ergs. In 6, $V$ at $B = -5\cdot24$ and $Q = 1$, $\therefore E = 5/48 =$ ·104. The charge on the earthed sphere is neglected, for the potential is zero. In 7, $V$ at $B$ or $C = -\{5/24 + 5/74 - 1/14\}$ and $E =$ ·204. In 8, $E = 5\cdot24 - 5\cdot74 + 1\cdot14 =$ ·212. In 9, $E =$ ·053, ·184, ·212. In 10, $E = 3/32 =$ ·094.

§ 56.—**Ex. 3.** 139. **4.** 46·2. **5.** Of course the logarithm is Napierian, and = common logarithm × 2·303. The result is $= 40/\log 1·1 = 420$. **7.** $4\log\{(ac - ax)\,bx\}$, where $x$ is given by the quadratic in § 45. This

$$= 4\log\{c^2 - a^2 - b^2 + \sqrt{(c^2 - a^2 - b)^2 - 4a^2b^2}\}\,2abc.$$

§ 63.—**Ex. 2.** ·24. **3.** 217.

§ 89.—**Ex. 4.** 1·46. **5.** 2·1.

§ 94.—**Ex. 4.** Molecular wt. = 111, hence molecular conductivity = ·0643 × 111 ÷ 50 = ·143. This is enormous compared to data in previous questions; we infer it is in practical units and must be reduced to absolute units by multiplying by $10^{-9}$. Hence the specific conductivity = $1430 \times 10^{-13}$. This = $2 \times 10^{-3} \times q(u_0 + v_0)/F$, $\therefore\ u_0 + v_0$ = ·00074. **5.** ·218 (practical unit). Divide by atomic weight of NaCl; we get ·0037.

§ 105.—**Ex. 6.** 2·5 volts. **7.** 1·14 volts.

§ 110.—**Ex. 9.** 28·6. **10.** 171 ohms. **11.** ·3 amp. **12.** ·445 amp.

§ 124.—**Ex. 4.** 6·29 ohms; 95·9 microhms.

§ 131.—**Ex. 4.** 1·62 volts, $3\frac{1}{3}$ ohms.

§ 132.—**Ex. 4.** 11·7 ohms, 16·2 c.cm. **5.** 10,000 ohms.

§ 138.—**Ex. 2.** 1·59 cm. **3.** 7·999 gm.-wt., = 7847 dynes.

§ 140.—**Ex. 2.** Fields = 1·13, ·29; Potentials = 0, 2·21.

§ 143.—**Ex. 1.** 37·2 cm. dynes, 87·6 cm. dynes.

§ 145.—**Ex.** 17·32.

§ 150.—**Ex. 10.** $X = 2M \cos\theta/CO^3$. **11.** If $F = M/CO^3$, the formula is $\tan\phi = 2F\cos\theta/(H + F\sin\theta)$. If $H > F$, $\phi$ never can = 90°. **14.** 14·4. At nearly 42° 40′ to meridian. **15.** 1840.

§ 157.—**Ex. 12.** ·0080 amp. **13.** ·00012 amp. **14.** 3 amp., 53 amp. **15.** ·31 amp. **16.** 10 turns. **17.** 1·32. **18.** ·145, 12 turns.

§ 167.—**Ex. 3.** Potential = $2\pi c\,(1 - z/\sqrt{z^2 + r^2})$. **4.** Solid angle = $2\pi/3$; $\therefore$ potential = $2\pi c/3$. **6.** Let $n$ = number of sides, $a$ = each side, $r$ = distance of point considered from each corner, $c$ = current. Then the potential

$$= 2c\left\{\pi - n\cos^{-1}\left(\sqrt{r^2 - \tfrac{1}{4}a^2}/r\,\cos\frac{\pi}{n}\right)\right\}.$$

§ 173.—**Ex. 7.** 18·1, (1) 10·1, (2) 5·0, (3) ·0015, ·00073.

§ 175.—**Ex. 4.** ·0111 amp. **5.** A fourth as great, ·00283. **6.** ·074 amp.

§ 178.—**Ex.** 493 cm. dynes.

§ 185.—**Ex.** ·159 : 1.

§ 187.—**Ex. 5.** ·154 per sec., ·308 per period. $8\frac{1}{2}$ sec. from initial disturbance. 11·3. **6.** Period multiplied by 1·013.

§ 188.—**Ex. 6.** ·86 M.F.

§ 190.—**Ex. 7.** Divided by 5·7. **8.** Kicks would be equal. **9.** Nearly 1,500,000.

§ 197.—**Ex. 5.** ·227 volt.

§ 198.—**Ex. 6.** $1{\cdot}12 \times 10^{-5}$, $3{\cdot}95 \times 10^{-7}$. **9.** ·12 volt. **10.** ·0314 volt, ·020 volt, ·0222 volt.

§ 201.—**Ex. 7.** $3{\cdot}17 \times 10^{-4}$.

§ 204.—**Ex. 3.** $1{\cdot}6 \times 10^{-3}$ coulombs. **4.** $3{\cdot}17 \times 10^{-4}$, $6{\cdot}34 \times 10^{-4}$. **5.** Multiplied by ·195.

§ 211.—**Ex. 5.** ·00025 Joule, ·00006 calorie.

§ 227.—**Ex. 9.** $2\pi cr^2\mu/a$.

§ 230.—**Ex. 6.** $2\pi n_1 n_2 N r^2 \mu\, b$, if $r$ be radius of the section of iron wire (absolute units). **7.** $2\mu b l n_1 n_2/a$, $2\mu b l n_1{}^2/a$, $2\mu b l n_2{}^2/a$. **8.** $1{\cdot}05 \times 10^{-5}$ Henries.

§ 247.—**Ex. 3.** If unit current be an ampere, the heats produced per second in ergs are 37900, − 43600, 3100, 15700. **4.** $5{\cdot}0 \times 10^{-5}$. **5.** ·00301. **6.** 64·7 at − 95°; $64{\cdot}9 + {\cdot}002t$. **7.** $1{\cdot}377 + {\cdot}000388t$.

§ 272.—**Ex. 4.** 306 E.M. units; $9{\cdot}2 \times 10^{12}$ E.S. units. **5.** 2·56. **6.** $x \propto v^2$ $y \propto v$; $\therefore$ $x \propto y^2$, and locus is parabolic.

§ 282.—**Ex. 3.** Mass = $6{\cdot}11 \times 10^{-12}$, field 9·8. **4.** $6{\cdot}16 \times 10^{-4}$. **5.** As 1 to 870. **6.** $3{\cdot}15 \times 10^{-6}$; $12{\cdot}6 \times 10^{-6}$.

§ 296.—**Ex. 3.** 530,000 yr.; 1/220 gm. **4.** 1·38000. **5.** $5E/226 \times 1700$ where $E = 2{\cdot}9 \times 10^{14}$. This = $3{\cdot}9 \times 10^{9}$ E.S. units per year. **6.** $6{\cdot}4 \times 10^{10}$ calories per sec., $= 2{\cdot}7 \times 10^{17}$ ergs per sec., per 1 gm. The mean velocity is about $2{\cdot}3 \times 10^{5}$.

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